Analytical formulation of a discrete chiral elastic metamaterial model

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ABSTRACT

By embedding appropriately designed chiral local resonators in a host elastic media, a chiral metamaterial with simultaneously negative effective density and bulk modulus can be achieved. In this work, an two dimentional (2D) ideal discrete model for the chiral elastic metamaterial is proposed. The discrete dynamic equation is derived and then homogenized to give the continuous description of the metamaterial. The homogenization procedure is validated by the agreement of the dispersion curve of the discrete and homogenized formulations. The form of homogenized governing equations of the metamaterial cannot be classified as a traditional Cauchy elastic theory. This result conforms the conscience that the Cauchy elasticity cannot reflect the chirality, which is usually captured by higher order theory such as the non-centrosymmetric micropolar elasticity. However when reduced to a (2D) problem, the existing chiral micropolar theory becomes non-chiral. Based on reinterpretation of isotropic tensors in a 2D case, we propose a continuum theory to model the chiral effect for 2D isotropic chiral solids. This 2D chiral micropolar theory constitutes a hopeful macroscopic framework for the theory development of chiral metamaterials.

Keywords: Chiral metamaterial, Discrete model, Chiral micropolar elasticity, Plane problem

1. INTRODUCTION

In 1967, Veselago theoretically investigated a visionary material with simultaneously negative permittivity and permeability [1]. This material is termed as a left-handed material (LHM). The concept did not become a reality until 2001 when Shelby et al [2] proposed designs of structured materials with microstructures of metallic wires and split-ring resonators which exhibit separately negative permittivity and permeability. The LHM leads to many unusual characteristics for applications in information and communication technologies.

In the same time, there has been also a great interest to design elastic and acoustic (EA) metamaterials. The EA metamaterial with negative mass density (NMD) was experimentally demonstrated through the localized resonance structure constructed by coating a heavy sphere with soft silicone rubber which is then encased in epoxy [3]. On the other hand, Fang et al. [4] designed an acoustic metamaterial with negative bulk modulus (NBM) by using an array of subwavelength Helmholtz resonators. Ding et al [5] proposed a double-negative EA metamaterial by combining a double unit structure with an array of bubble-contained water spheres and an array of rubber-coated gold spheres in epoxy matrix. Similarly, various EA metamaterial designs of engineered solid and fluid unit mixture were proposed to obtain simultaneously NMD and NBM. However, the aforementioned designs all require the combined use of solid and fluid media and are difficult to fabricate. Recently Lai et al. [6] proposed a design of four-components metamaterial consisting

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of solid materials which can achieve triply negative properties by exiting monopolar, dipolar and quadrupolar resonance. In a previous work [7], we proposed a practical doubly negative elastic metamaterial design which completely make use of solid materials by using chirality. The metemeterial pattern and its unit cell are shown in Fig. 1. The unit cell is based on the 2D analogy of the well-known three-component sonic crystal [3]. A number of (n_s) slots with width t_s are cut out from the coating material. The slots are equi-spaced in azimuth and oriented at an angle θ_s with respect to the radial direction. It is known that the traditional three-component metamaterial without the slots gives NMD because of the translational resonance of the core. The chirality introduces additionally the coupling of bulk deformation with the core rotation, hence the simultaneously NMD and NBM are achieved when the translational and rotational resonance is excited at overlapped frequency range.

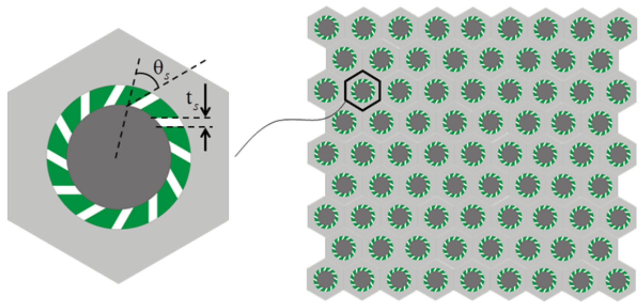

Fig. 1.

The effective (homogenized) properties of the chiral metamaterial are formulated within the classical Cauchy elastic theory. Despite the predicted effective NMD and NBN conforms well with the dispersion relation and successfully explained the left-handed wave phenomenon , e.g. the negative refraction, some unusual features of this chiral metamaterial are beyond the classical elasticity. For instance, we have found that, in the doubly negative frequency band, the wave mode is of mixed type, which means that the particle polarization is neither longitudinal (P) nor transverse (S). This contradict with the fact that the triangular lattice homogenize to an isotropic material at long wave limit. When the resonance of the core is absent, the wave modes are almost of pure P or S wave types. However, as seen in Fig.1, the geometrical complexity of the continuum based model is difficult to give an analytical insight for the chiral metamaterial.

In continuum mechanics, chirality is considered in the context of generalized elasticity, e.g. micropolar (Cosserat) theory [8,9]. A general isotropic chiral (also known as non-centrosymmetric, acentric or hemitropic) micropolar theory introduces three additional material constants compared to the non-chiral theory, the additional material parameters change their signs according to the handedness of the microstructure to represent the effect of chirality [10-14]. This theory provides an efficient tool for modeling the chiral effect presented in materials and structures, e.g., loading transfer in carbon nanotubes and chiral rods [15], mechanics of bone [16] and wave propagation in chiral solids [17]. However, when this theory is applied to a planar isotropic case, the variables describing the chiral effect disappear and the resulting theory becomes basically non-chiral [18]. Therefore the basic characteristic of a planar chiral solid cannot be properly modeled by the existing theory. Therefore, for whole type of such 2D chiral metamaterials, there is no an appropriate theory to characterize their chiral effects.

In this paper, based on the practical continuum based chiral metamaterial model, we propose a ideal discrete chiral metamaterial model which can represent the fundamental feature of the continuum analog. The discrete model can be analytically dealt with to discover some unique properties of the chiral metamaterial. The discrete dyanamic equations of the model are derived and then homogenized to give the continuous macroscopic description. On the other hand, based on reinterpretation of isotropic tensors in a 2D case, we propose a new type of micropolar constitutive equation to characterize the chirality for 2D solids. As will be shown, the 2D chiral micropolar theory can cover the main feature of the chiral metamaterial and can serve as a macroscopic framework for this type of metamaterial.

2. ANALYSIS OF DISCRETE CHIRAL METAMATERIAL MODEL

In line with the continuum based model in Fig.1, a ideal discrete model is proposed and depicted in Fig. 2. The host media is mimicked by a square mass spring system. The elastic properties of the host is represented by the massless spring with longitudinal and shear spring constants K and Γ, and the mass M is concentrated at the cross point. The chiral resonator consists of a square with mass m and rotational inertia j and four weak springs with only longitudinal spring constant k. Within each unit cell, the four weak springs connect the four corners of the square core with the cross point of the host. As shown in Fig.2, the orientation of the core forms a certain angle θ with the host lattice. It is just this angle produce the chirality of the model, when $\theta = 0$, the chirality will be absent. The weak springs serve as the soft coating material, and at certain frequency they together with the heavy square core can create couple translational and rotational resonance. The host lattice constant is a, and the edge length of the square core is b.

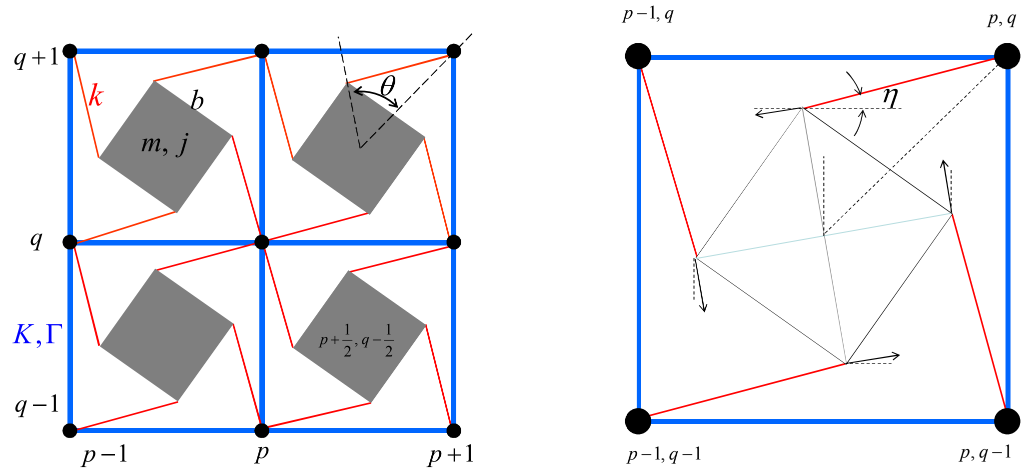

Fig. 2.

For the convenience of formulation, the several geometric identities is defined:

$$A_c = \frac{\sqrt{2}}{2} b \cos\left(\frac{\pi}{4} + \theta - \frac{\pi}{2}\right), \quad A_s = \frac{\sqrt{2}}{2} b \sin\left(\frac{\pi}{4} + \theta - \frac{\pi}{2}\right), \tag{1}$$

$$\eta = \frac{\pi}{4} - \arctan\left(\frac{b\sin\theta}{a - b\cos\theta}\right), \tag{2}$$

$$C = \cos\eta, \qquad S = \sin\eta, \tag{3}$$

where the angle η is the angle between the upper weak spring with the horizontal, see the unit cell shown in Fig. 2b. We use p and q as the index of the host mass points, and p' and q' as the index of the square cores, respectively. The index of the unit cell is define by the index of upper-right corner host mass. With this rule, for the square in the unit cell (p , q), we have $p' = p - 1/2$ and $q' = q - 1/2$.

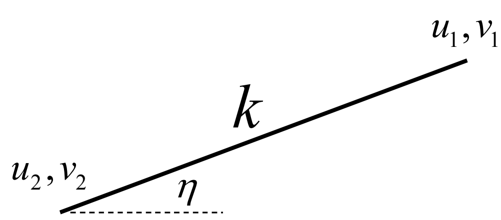

Fig. 3.

For a single weak spring with orientation angle η as shown in Fig.3, the deformation energy is given by its end point displacements

$$V = \frac{1}{2} k \left[C^2 (u_1 - u_2)^2 + S^2 (v_1 - v_2)^2 + 2CS(u_1 - u_2)(v_1 - v_2) \right]. \tag{4}$$

Then the the total energy of a single unit cell can be obtained and summed up to give the Hamiltonian of the system

$$\begin{aligned} 2H = \sum_{p,q} \Big\{ & M\left(\dot{u}_{p,q}^2 + \dot{v}_{p,q}^2\right) + m\left(\dot{u}_{p',q'}^2 + \dot{v}_{p',q'}^2\right) + j\dot{\phi}_{p',q'}^2 \\ & + K\left(u_{p,q} - u_{p-1,q}\right)^2 + \Gamma\left(v_{p,q} - v_{p-1,q}\right)^2 + K\left(v_{p,q} - v_{p,q-1}\right)^2 + \Gamma\left(u_{p,q} - u_{p,q-1}\right)^2 \\ & + kC^2\left[u_{p,q} - \left(u_{p',q'} - \phi A_c\right)\right]^2 + kS^2\left[v_{p,q} - \left(v_{p',q'} - \phi A_s\right)\right]^2 \\ & + 2kCS\left[u_{p,q} - \left(u_{p',q'} - \phi A_c\right)\right]\left[v_{p,q} - \left(v_{p',q'} - \phi A_s\right)\right] \\ & + kS^2\left[u_{p-1,q} - \left(u_{p',q'} + \phi A_s\right)\right]^2 + kC^2\left[v_{p-1,q} - \left(v_{p',q'} - \phi A_c\right)\right]^2 \\ & - 2kCS\left[u_{p-1,q} - \left(u_{p',q'} + \phi A_s\right)\right]\left[v_{p-1,q} - \left(v_{p',q'} - \phi A_c\right)\right] \\ & + kC^2\left[u_{p-1,q-1} - \left(u_{p',q'} + \phi A_c\right)\right]^2 + kS^2\left[v_{p-1,q-1} - \left(v_{p',q'} + \phi A_s\right)\right]^2 \\ & + 2kCS\left[u_{p-1,q-1} - \left(u_{p',q'} + \phi A_c\right)\right]\left[v_{p-1,q-1} - \left(v_{p',q'} + \phi A_s\right)\right] \\ & + kS^2\left[u_{p,q-1} - \left(u_{p',q'} - \phi A_s\right)\right]^2 + kC^2\left[v_{p,q-1} - \left(v_{p',q'} + \phi A_c\right)\right]^2 \\ & - 2kCS\left[u_{p,q-1} - \left(u_{p',q'} - \phi A_s\right)\right]\left[v_{p,q-1} - \left(v_{p',q'} + \phi A_c\right)\right] \Big\} \end{aligned} , \tag{5}$$

where u , v and ϕ denote the two displacement and the rotational degree of freedom, respectively. Hence the discrete dynamic equations for the host mass and the square mass can be easily established by using the Hamilton's principle

$$\begin{aligned} M\omega^2 u_{p,q} = & K\left(2u_{p,q} - u_{p-1,q} - u_{p+1,q}\right) + \Gamma\left(2u_{p,q} - u_{p,q-1} - u_{p,q+1}\right) + 2ku_{p,q} \\ & - kC^2\left(u_{p',q'} + u_{p'+1,q'+1}\right) - kS^2\left(u_{p'+1,q'} + u_{p',q'+1}\right) + kCS\left(v_{p'+1,q'} + v_{p',q'+1} - v_{p',q'} - v_{p'+1,q'+1}\right), \\ & + \left(kC^2 A_c + kCSA_s\right)\left(\phi_{p',q'} - \phi_{p'+1,q'+1}\right) + \left(kS^2 A_s + kCSA_c\right)\left(\phi_{p',q'+1} - \phi_{p'+1,q'}\right) \end{aligned} \tag{6}$$

$$\begin{aligned} M\omega^2 v_{p,q} = & K\left(2v_{p,q} - v_{p,q-1} - v_{p,q+1}\right) + \Gamma\left(2v_{p,q} - v_{p-1,q} - v_{p+1,q}\right) + 2kv_{p,q} \\ & - kC^2\left(v_{p',q'+1} + v_{p'+1,q'}\right) - kS^2\left(v_{p',q'} + v_{p'+1,q'+1}\right) + kCS\left(u_{p'+1,q'} + u_{p',q'+1} - u_{p',q'} - u_{p'+1,q'+1}\right), \\ & + \left(kC^2 A_c + kCSA_s\right)\left(\phi_{p'+1,q'} - \phi_{p',q'+1}\right) + \left(kS^2 A_s + kCSA_c\right)\left(\phi_{p',q'} - \phi_{p'+1,q'+1}\right) \end{aligned} \tag{7}$$

$$m\omega^2 u_{p',q'} = 2ku_{p',q'} - kC^2\left(u_{p,q} + u_{p-1,q-1}\right) - kS^2\left(u_{p-1,q} + u_{p,q-1}\right) + kCS\left(v_{p-1,q} + v_{p,q-1} - v_{p,q} - v_{p-1,q-1}\right), \tag{8}$$

$$m\omega^2 v_{p',q'} = 2kv_{p',q'} - kC^2\left(v_{p-1,q} + v_{p,q-1}\right) - kS^2\left(v_{p,q} + v_{p-1,q-1}\right) + kCS\left(u_{p-1,q} + u_{p,q-1} - u_{p,q} - u_{p-1,q-1}\right), \tag{9}$$

$$
\begin{aligned}
j\omega^2\phi_{p',q'} &= 4k\left(CA_c+SA_s\right)^2\phi_{p',q'} \\
&+kC^2A_c\left(u_{p,q}-u_{p-1,q-1}+v_{p-1,q}-v_{p,q-1}\right)+kS^2A_s\left(u_{p,q-1}-u_{p-1,q}+v_{p,q}-v_{p-1,q-1}\right). \\
&+kCSA_c\left(v_{p,q}-v_{p-1,q-1}+u_{p,q-1}-u_{p-1,q}\right)+kCSA_s\left(v_{p-1,q}-v_{p,q-1}+u_{p,q}-u_{p-1,q-1}\right)
\end{aligned}
\tag{10}
$$

In above equations time harmonic variation of the variants is assumed. Note that because of the chiral pattern of the weak springs, they provide the rotational stiffness to the relative rotation between the square core and the host, as seen in Eq.(10).

To get a continuum description, Eq.(6-10) can be homogenized by replace the difference relation by the Taylor's series expansion at long wave limit. Obviously, if we describe the chiral metamaterial model in such a detail as Eq.(6-10) then no negative properties would be obtained, since both the host and local variables are present. The exotic properties of metamaterial originates from that macroscopically we measure only the physical quantities belongs to the host and treat the local resonators as 'hiden', such as the discrete mass-in-mass model used to clarify the NMD [milton]. To this end, before the homogenization procedure using Taylor's expansion, we must retain only the discrete equations (6) and (7) related to the host, and represent variables related to the square core in these two equations by the host variables. This can be done by solving $u_{p',q'}$, $v_{p',q'}$ and $\phi_{p',q'}$ from Eq.(8-10) and substituting into Eq.(6) and (7). The process is tedious by straightforward, and we show only the resulting discrete equation about $u_{p,q}$ for example

$$
\begin{aligned}
M\omega^2u_{p,q} &= K\left(2u_{p,q}-u_{p-1,q}-u_{p+1,q}\right)+\Gamma\left(2u_{p,q}-u_{p,q-1}-u_{p,q+1}\right)+2ku_{p,q} \\
&+\frac{k^2}{m\omega^2-2k}\Big\{S^2\left(u_{p-1,q+1}+2u_{p,q}+u_{p+1,q-1}\right)+C^2\left(u_{p-1,q-1}+2u_{p,q}+u_{p+1,q+1}\right) \\
&+CS\left(v_{p-1,q-1}-v_{p-1,q+1}-v_{p+1,q-1}+v_{p+1,q+1}\right)\Big\} \\
&+\frac{k^2\left(CA_c+SA_s\right)^2}{4k\left(CA_c+SA_s\right)^2-j\omega^2}\Big\{C^2\left(u_{p-1,q-1}-2u_{p,q}+u_{p+1,q+1}-v_{p-1,q}+v_{p,q-1}+v_{p,q+1}-v_{p+1,q}\right) \\
&+S^2\left(u_{p-1,q+1}-2u_{p,q}+u_{p+1,q-1}+v_{p-1,q}-v_{p,q-1}-v_{p,q+1}+v_{p+1,q}\right) \\
&+CS\left(2u_{p-1,q}-2u_{p,q-1}-2u_{p,q+1}+2u_{p+1,q}+v_{p-1,q-1}-v_{p-1,q+1}-v_{p+1,q-1}+v_{p+1,q+1}\right)\Big\}
\end{aligned}
\tag{11}
$$

Note on the right hand side of the above equation, there is two items relevant with the resonance. Under time harmonic variation, the translational motion of the square core takes effect by the term with coefficient $k^2/(m\omega^2-2k)$, while the rotational motion of the square core takes effect by the term $k^2\left(CA_c+SA_s\right)^2/[4k\left(CA_c+SA_s\right)^2-j\omega^2]$. The singularity presenting in these two items clearly shows that negative effective constants is possible under resonance. By utilizing the Taylor's expansion

$$
\begin{aligned}
&u_{p,q}=u \\
&u_{p\pm1,q\pm1}=u+u_{,x}dx_{\pm1}+u_{,y}dy_{\pm1}+\frac{1}{2}u_{,xx}dx_{\pm1}^2+\frac{1}{2}u_{,yy}dy_{\pm1}^2+f_{xy}dx_{\pm1}dy_{\pm1}, \\
&dx_{\pm1}=dy_{\pm1}=\pm a
\end{aligned}
\tag{12}
$$

(similar for v and ϕ, a comma preceding the subscript means the partial differential to that coordinate) and retaining the leading order terms for the same coeffecients, we obtain the homogenized governing equations regarding to the degree of freedom of the host

$$
\begin{aligned}
\frac{M_{eff}}{a^2}\ddot{u} &= [K-T(C+S)^2]u_{xx}+[\Gamma+T(C-S)^2]u_{yy}-4CSTv_{xy} \\
&+T(C^2-S^2)v_{xx}-T(C^2-S^2)v_{yy}-2T(C^2-S^2)u_{xy}
\end{aligned}
\tag{13}
$$

$$\frac{M_{eff}}{a^2}\ddot{v} = [\Gamma - T(C-S)^2]v_{xx} + [\mathrm{K} - T(C+S)^2]v_{yy} - 4CSTu_{xy}$$
$$+T(C^2 - S^2)u_{xx} - T(C^2 - S^2)u_{yy} + 2T(C^2 - S^2)v_{xy} \tag{14}$$

where

$$M_{eff} = M - \frac{2km}{m\omega^2 - 2k}, \tag{15}$$

is the effective mass of a unit cell, and

$$T = \frac{k^2\left(CA_c + SA_s\right)^2}{4k\left(CA_c + SA_s\right)^2 - j\omega^2}. \tag{16}$$

The coefficients on the right hand side of Eq.(13) and (14) can be interpreted as the effective stiffness of the homogenized media. Note when $\theta = 0$,

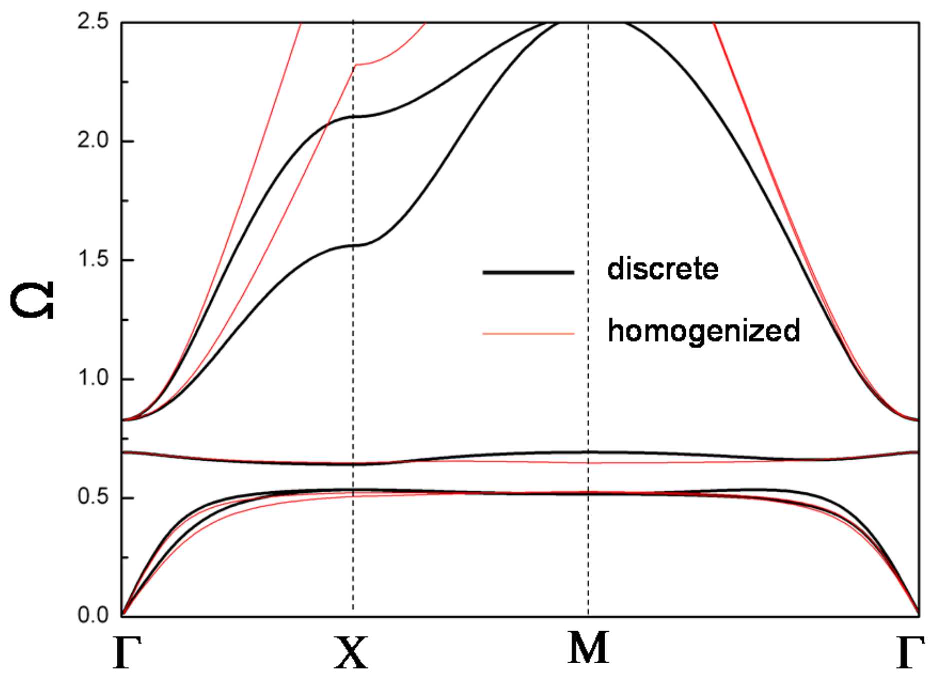

Fig. 4.

3. PLANAR ISOTROPIC MICROPOLAR MODEL WITH CHIRALITY

Characterization of material chirality is closely related to the concept of pseudo (or axial) tensors, they alternate the sign with a mirror reflecting transformation or the handedness change of the underlying coordinate system, and ordinary (or polar) tensors are not affected by such actions [19]. Both types of tensors coexist in various elastic formulations, but strain energy density must be independent of handedness.

Classical elasticity theory excludes chirality [11], since in the energy density

$$w = \frac{1}{2}\varepsilon_{ij}C_{ijkl}\varepsilon_{kl}, \tag{1}$$

the strain ε is an ordinary tensor. To maintain w as an ordinary scalar, the elastic tensor C cannot be pseudo. So to include chirality, micropolar theory [20] is considered in this paper. In the micropolar theory, rotational degree of freedom (DOF) ϕ_i is introduced in addition to the displacement u_i on a material point. The strain and curvature play as deformation measures

$$\varepsilon_{kl} = u_{l,k} + e_{lkm}\phi_m, \tag{2a}$$

$$\kappa_{kl} = \phi_{k,l}, \tag{2b}$$

and the balance of stress σ_{ji} and couple stress m_{ji} are governed by

$$\sigma_{ji,j} = \rho\partial^2 u_i / \partial t^2, \tag{3a}$$

$$m_{ji,j} + e_{ikl}\sigma_{kl} = j\partial^2\phi_i / \partial t^2 \quad , \tag{3b}$$

where e_{ijk} is the Levi-Civita tensor, ρ and j are the density and micro-inertia, respectively. In this paper, the Einstein's summation convention is used and the comma in subscript denotes partial differentiation with respect to spatial coordinates. The strain energy density for a general linear elastic micropolar media is expressed as a quadratic form in terms of asymmetric strain and curvature tensors

$$w = \frac{1}{2}\varepsilon_{ij}C_{ijkl}\varepsilon_{kl} + \frac{1}{2}\kappa_{ij}D_{ijkl}\kappa_{kl} + \varepsilon_{ij}B_{ijkl}\kappa_{kl} \quad , \tag{4}$$

where C, D and B are elastic tensors of rank four. Then the constitutive relation can be derived as

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} + B_{ijkl}\phi_{k,l}, \tag{5a}$$

$$m_{ij} = B_{ijkl}\varepsilon_{kl} + D_{ijkl}\phi_{k,l}. \tag{5b}$$

It should be noted that the microrotation vector ϕ_i and curvature tensor κ_{ij} are pseudo, thus in Eq.(4), B_{ijkl} must be a pseudo tensor and thus represent the chirality. A micropolar solid with $B_{ijkl} \neq 0$ is usually referred as non-centrosymmetric. Let us consider the isotropic case and focus only the chiral part of Eq.(5). A general isotropic tensor of rank four takes the form as

$$B_{ijkl} = B_1\delta_{ij}\delta_{kl} + B_2\delta_{ik}\delta_{jl} + B_3\delta_{il}\delta_{jk}, \tag{6}$$

where δ_{jk} is Kronecker delta. The chiral part of Eq.(5) then reads

$$\sigma_{ij} = B_1\delta_{ij}\phi_{k,k} + B_2\phi_{i,j} + B_3\phi_{j,i}, \tag{7a}$$

$$m_{ij} = B_1\delta_{ij}\varepsilon_{k,k} + B_2\varepsilon_{i,j} + B_3\varepsilon_{j,i}. \tag{7b}$$

A planar micropolar problem in $x_1 - x_2$ plane is defined by $u_3 = \phi_1 = \phi_2 = \partial / \partial x_3 = 0$, while the non zero quantities are u_α, ϕ_3, $\phi_{3,\alpha}$, $\varepsilon_{\alpha\beta}$, $\sigma_{\alpha\beta}$ and $m_{\alpha 3}$, respectively, with Greek subscripts ranging from 1 to 2. Specialized to the 2D case, it is easy to verify that Eq.(7) is trivial, as a result, chirality represented by the isotropic B_{ijkl} disappears. However, for 2D isotropic chiral solids, e.g., planar triangular chiral lattices, chirality is a basic feature of such structures and should be characterized by a correct constitutive modeling. Therefore there should be something missing in Eqs. (5) and (6) when reduced to the 2D case.

To circumvent this problem, we first discuss some basic properties of isotropic tensors. The basic forms of isotropic tensors of rank 0, 2 and 3 are just the scalar, Kronecker delta δ_{jk} and Levi-Civita tensor e_{ijk}, respectively. A vector which is rank one cannot be isotropic, and e_{ijk} is a pseudo tensor. Any isotropic tensor with rank greater than three can be constructed by scalar, δ_{jk} and e_{ijk}, just as Eq.(6) for an isotropic tensor of rank four. However, in the 2D case, the Levi-Civita tensor e_{ijk} is restricted to the form $e_{ijk} \equiv e_{3\alpha\beta}$, which is in-plane isotropic and equivalent to an isotropic tensor of rank two. For the same reason, there is no in-plane third order isotropic tensor in the current 2D situation. The isotropic B tensor also vanishes, since the energy density of the 2D case can be rewritten as

$$w = \frac{1}{2}\varepsilon_{\alpha\beta}C_{\alpha\beta\gamma\rho}\varepsilon_{\gamma\rho} + \frac{1}{2}\phi_{3,\alpha}D_{\alpha\beta}\phi_{3,\beta} + \varepsilon_{\alpha\beta}B_{\alpha\beta\gamma}\phi_{3,\gamma}, \tag{8}$$

where B reduces to the tensor of rank three and cannot be isotropic except zero. However, with $e_{3\alpha\beta}$ and $\delta_{\alpha\beta}$, we thus have more choices to construct a 2D isotropic tensor of rank four. In particular, we have

$$\bar{C}_{\alpha\beta\gamma\rho} = \bar{C}_1\delta_{\alpha\beta}\delta_{\gamma\rho} + \bar{C}_2\delta_{\alpha\gamma}\delta_{\beta\rho} + \bar{C}_3\delta_{\alpha\rho}\delta_{\beta\gamma} \tag{9a}$$

$$\tilde{C}_{\alpha\beta\gamma\rho} = \tilde{C}_1\delta_{\alpha\beta}e_{3\gamma\rho} + \tilde{C}_2\delta_{\gamma\rho}e_{3\alpha\beta} + \tilde{C}_3\delta_{\alpha\gamma}e_{3\beta\rho} + \tilde{C}_4\delta_{\beta\rho}e_{3\alpha\gamma} + \tilde{C}_5\delta_{\alpha\rho}e_{3\beta\gamma} + \tilde{C}_6\delta_{\beta\gamma}e_{3\alpha\rho} \tag{9b}$$

$$\hat{C}_{\alpha\beta\gamma\rho} = \hat{C}_1 e_{3\alpha\beta}e_{3\gamma\rho} + \hat{C}_2 e_{3\alpha\gamma}e_{3\beta\rho} + \hat{C}_3 e_{3\alpha\rho}e_{3\beta\gamma} \tag{9c}$$

Eq.(9a) is just the 2D version of Eq.(6). By utilizing the identity $e_{3\alpha\beta}e_{3\gamma\rho} = \delta_{\alpha\gamma}\delta_{\beta\rho} - \delta_{\alpha\rho}\delta_{\beta\gamma}$, Eq.(9c) is found to take the same form as Eq.(9a), this is not surprising since $\hat{C}_{\alpha\beta\gamma\rho}$ is the product of two pseudo tensors, hence it is an ordinary tensor. In summary, we conclude that for a 2D micropoar problem the generic form of an isotropic tensor of rank four can be given by

$$C_{\alpha\beta\gamma\rho} = \bar{C}_{\alpha\beta\gamma\rho} + \tilde{C}_{\alpha\beta\gamma\rho} \tag{10}$$

Reexamining the energy density in Eq.(8) with the help of Eq.(10) and $D_{\alpha\beta} = D_1\delta_{\alpha\beta}$, we have

$$w = \tfrac{1}{2}\varepsilon_{\alpha\beta}\bar{C}_{\alpha\beta\gamma\rho}\varepsilon_{\gamma\rho} + (\tilde{C}_1 + \tilde{C}_2)\varepsilon_{\alpha\alpha}e_{3\gamma\rho}\varepsilon_{\gamma\rho} + \tfrac{1}{2}D_1\phi_{3,\alpha}\delta_{\alpha\beta}\phi_{3,\beta} \tag{11}$$

Introducing the Lame's constant λ, μ, antisymmetric shear modulus κ, higher order modulus γ, and a single chiral parameter $2A \equiv \tilde{C}_1 + \tilde{C}_2$, the in-plane isotropic micropolar elastic tensors with chirality are thus written as

$$\bar{C}_{\alpha\beta\gamma\rho} = \lambda\delta_{\alpha\beta}\delta_{\gamma\rho} + (\mu+\kappa)\delta_{\alpha\gamma}\delta_{\beta\rho} + (\mu-\kappa)\delta_{\alpha\rho}\delta_{\beta\gamma} \tag{12a}$$

$$\tilde{C}_{\alpha\beta\gamma\rho} = A(\delta_{\alpha\beta}e_{3\gamma\rho} + \delta_{\gamma\rho}e_{3\alpha\beta}) \tag{12b}$$

$$\bar{D}_{\alpha\beta} = \gamma\delta_{\alpha\beta} \tag{12c}$$

Note that Eq.(12b) has a symmetric form conforming with the requirement of major symmetry for the C tensor. The constitutive equation then becomes

$$\sigma_{\alpha\beta} = \lambda\delta_{\alpha\beta}\varepsilon_{\rho\rho} + (\mu+\kappa)\varepsilon_{\alpha\beta} + (\mu-\kappa)\varepsilon_{\beta\alpha} + A\delta_{\alpha\beta}e_{3\gamma\rho}\varepsilon_{\gamma\rho} + Ae_{3\alpha\beta}\varepsilon_{\rho\rho}, \tag{13}$$

$$m_{\alpha 3} = \gamma\phi_{3,\alpha}.$$

It is interesting to note that the pseudo tensor $\tilde{C}_{\alpha\beta\gamma\rho}$ representing the chirality relates to both the normal stress and normal strain, this is different from the B_{ijkl} tensor for a 3D case. The physical meaning of $\tilde{C}_{\alpha\beta\gamma\rho}$ is however very clear. Consider the relevant part in Eq.(11)

$$A(\delta_{\alpha\beta}e_{3\gamma\rho} + \delta_{\gamma\rho}e_{3\alpha\beta})\varepsilon_{\alpha\beta}\varepsilon_{\gamma\rho} = 2A\varepsilon_{\rho\rho}(e_{3\alpha\beta}\varepsilon_{\alpha\beta}), \tag{14}$$

the spherical strain $\varepsilon_{\rho\rho}$ represents the bulk deformation at a material point, it is obviously independent of the handedness of the frame. On the other hand, $e_{3\alpha\beta}\varepsilon_{\alpha\beta} = -2(\phi_3 - \psi_3)$ is the pure rotation of a point, with $\psi_3 = e_{3\alpha\beta}u_{\beta,\alpha}/2$ denoting the macro rigid rotation, therefore, it is an axial quantity depending on the handedness. This chiral term in the energy density clearly demonstrates that a pure rotation can produce shrink or dilatation of the material, and vice versa. This mechanism derived from a continuum formulation explains correctly the behavior of a real 2D chiral structure(e.g. the triangular chiral lattice schematically depicted in Fig. 1), which will be discussed in detail in the following section. This is also the unique mechanism of the chiral lattice to produce the negative Poisson's ratio.

The constitutive law of Eq. (13) can be rearranged in a matrix form as

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{21} \\ m_{13} \\ m_{23} \end{Bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & -A & A & 0 & 0 \\ \lambda & 2\mu+\lambda & -A & A & 0 & 0 \\ -A & -A & \mu+\kappa & \mu-\kappa & 0 & 0 \\ A & A & \mu-\kappa & \mu+\kappa & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma \end{bmatrix} \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{2,1}-\phi \\ u_{1,2}+\phi \\ \phi_{,1} \\ \phi_{,2} \end{Bmatrix}. \tag{15}$$

It has four classical micropolar elastic constants and a new parameter A characterizing the chiral effect. When the handedness of the material pattern is flipped over, the chiral constant A should reverse its sign to maintain the invariance of the strain energy density, and the other constants remain unchanged:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{21} \\ m_{13} \\ m_{23} \end{Bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & A & -A & 0 & 0 \\ \lambda & 2\mu+\lambda & A & -A & 0 & 0 \\ A & A & \mu+\kappa & \mu-\kappa & 0 & 0 \\ -A & -A & \mu-\kappa & \mu+\kappa & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma \end{bmatrix} \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{2,1}-\phi \\ u_{1,2}+\phi \\ \phi_{,1} \\ \phi_{,2} \end{Bmatrix} \tag{16}$$

Finally, with the proposed constitutive equation, the governing equations expressed in the displacements u , v and the microrotation $\phi \equiv \phi_3$ read

$$\rho\frac{\partial^2 u}{\partial t^2} = (\lambda+2\mu)u_{,xx} + (\mu+\kappa)u_{,yy} + (\lambda+\mu-\kappa)v_{,xy} + 2\kappa\phi_{,y} - A(v_{,xx} - v_{,yy} - 2u_{,xy} - 2\phi_{,x}) \tag{17a}$$

$$\rho\frac{\partial^2 v}{\partial t^2} = (\mu+\kappa)v_{,xx} + (\lambda+2\mu)v_{,yy} + (\lambda+\mu-\kappa)u_{,xy} - 2\kappa\phi_{,x} - A(u_{,xx} - u_{,yy} + 2v_{,xy} - 2\phi_{,y}) \tag{17b}$$

$$j\frac{\partial^2 \phi}{\partial t^2} = \gamma(\phi_{,xx} + \phi_{,yy}) - 4\kappa\phi + 2\kappa(v_{,x} - u_{,y}) - 2A(u_{,x} + v_{,y}). \tag{17c}$$

The form of Eqs. (15-17) looks like those of an anisotropic medium, however, they are fundamentally different and cannot be covered by even the most general anisotropy without chirality, since the parameter A and its sign form a unique pattern in the constitutive matrix. They are indeed in-plane isotropic guaranteed by Eq. (12). It should be mentioned that, for a 2D isotropic micropolar solid, Eq.(12) is the only possible form to include chiralty.

The constraint conditions on the five material constants can be obtained by imposing positive definiteness of the strain energy density w . It is convenient to decompose the strain tensor into hydrostatic, deviatoric symmetric and antisymmetric parts as [21]

$$\varepsilon_{\alpha\beta} = \delta_{\alpha\beta}\bar{\varepsilon} + \varepsilon^{d}_{(\alpha\beta)} + \varepsilon_{<\alpha\beta>}, \tag{18}$$

where

$$\bar{\varepsilon} = \frac{1}{2}\varepsilon_{\alpha\alpha}, \tag{19a}$$

$$\varepsilon^{d}_{(\alpha\beta)} = \frac{1}{2}(\varepsilon_{\alpha\beta} + \varepsilon_{\beta\alpha}) - \delta_{\alpha\beta}\bar{\varepsilon}, \tag{19b}$$

$$\varepsilon_{<\alpha\beta>} = \frac{1}{2}(\varepsilon_{\alpha\beta} - \varepsilon_{\beta\alpha}) = e_{\beta\alpha 3}(\phi - \psi). \tag{19c}$$

Substituting Eqs.(18) and (12) into (11) yields

$$w = 2\bar{\kappa}\,\bar{\varepsilon}^2 + 2\mu\varepsilon^d_{(\alpha\beta)}\varepsilon^d_{(\alpha\beta)} + 2\kappa(\phi-\psi)^2 + 4A\bar{\varepsilon}(\phi-\psi) + \frac{1}{2}\gamma\phi_{,\alpha}\phi_{,\alpha}, \tag{20}$$

where $\bar{\kappa} = \lambda + \mu$ is defined as the 2D bulk (area) modulus. This equation yields the necessary and sufficient conditions for the positive definiteness of w. Besides the four conditions for a classical micropolar medium given in the literature

$$\bar{\kappa} = \lambda + \mu > 0,\ \mu > 0,\ \kappa > 0,\ \gamma > 0, \tag{21}$$

an additional condition

$$A^2 < (\lambda + \mu)\kappa, \tag{22}$$

is imposed on the chiral constant A. It can be either positive or negative, but its absolute value is bounded.

4. CONCLUDING REMARKS

In this work, a new chiral metacomposite is suggested by integrating a two-dimensional chiral lattice and a metamaterial inclusion for the low-frequency bandgaps. The matematerial inclusion, which is responsible for the local resonance, composes of a heavy core and a soft coating layer. The in-plane wave propagation in the metacomposite is studied through the finite element technique and Bloch's theorem to illustrate specific wave properties. Effective dynamic properties of the chiral metacomposite are determined to understand wave mechanism of the low-frequency bandgaps in the chiral metacomposite. Tuning of the resulting low-frequency bandgaps is then discussed by adjusting microstructure parameters of the metamaterial inclusion and lattice geometry. Specifically, a design of a metacomposite beam structure for broadband low-frequency vibration suppression is demonstrated.

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