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A NOTE ON MICROSTRUCTURAL INTERPRETATION OF THE MATERIAL CONSTANTS FOR COUPLE STRESS THEORY

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Introduction

Micromechanics, aiming at establishing the relationship between the microstructural parameters and thermo-mechanical properties for heterogeneous materials, is recognized with a rapid development[1-5]. With the help of Hill's condition and representative volume element(RVE) concept [6], and through some adequate localization and homogenization procedures, the relation between the average stress and strain over the RVE is interpreted as the macroscopic properties for the studied heterogeneous materials. Although it is a great success in the composite materials, this approach fails to predict the size effect of the reinforced phases as well-observed. To this end, the high order stress theories or high order mediums, based on the original idea of the brothers Cosserat[7], are proposed. By introducing length scale parameters in the constitutive relation, the theories can account reasonably for the observed size effect in the heterogeneous materials [8,9].

However, the parameters introduced for the size effect such as the length scale parameters in the gradient theory or coupled stress coefficients lack clear physical basis. In this paper, based on the thin laminate plate theory, we will try to give an attempt to provide the physical explanation for the new material parameters introduced in the coupled stress theory. The paper will be arranged as follows: the constitutive relations for the couple stress theory will be briefly reviewed, and its special form with a transverse symmetry will be given, and applied to describe the constitutive relations of a thin plate element. Their relations with microstructural parameters will be provided through a micro-macro homogenization procedure in a thin plate element (RVE).

Couple stress description for a thin plate element

The general constitutive relation for couple stress theory with centro-symmetry enables the strain to uncouple from the rotation at the same point. For the symmetric stress and strain, the traditional relation still holds:

$$\Sigma_{ij} = C_{ijkl} E_{kl} \tag{1}$$

In addition, for the same point , another equation is needed to relate the couple stress m_{ij} and the rotation vector Φ_k, which are related to the displacement field U_i by $\Phi_k = 1/2\varepsilon_{klm}U_{m,l}$, and this gives

$$m_{ij} = B_{ijkl}\Phi_{k,l} \tag{2}$$

m_{ij} is an asymmetric tensor and denotes the couple stress. This is a new relation compared to the classical continua. So in the couple stress theory, both eqs(1) and (2) are necessary to describe the stress and strain relation, in addition to the relation between the couple and curvature at the same point.

In a general case, the coefficients for the stress and strain satisfies $C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}$ with 21 independent constants; and $B_{ijkl} = B_{klij}$ with 45 independent constants. In the case of transverse symmetry, it can be shown that there are 5 independent constants for C_{ijkl} and 8 for B_{ijkl}. This relation is now applied for a thin plate element (with transverse symmetry x_3), which represents a material point in the couple stress theory.

In this case, the non-zero in-plane stress components are $\Sigma_{11}, \Sigma_{22}, \Sigma_{12}, m_{11}, m_{12}, m_{21}, m_{22}$. They are related to the strain and rotation by:

$$\begin{Bmatrix} \Sigma_{11} \\ \Sigma_{22} \\ \Sigma_{12} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 \\ \bar{Q}_{12} & \bar{Q}_{11} & 0 \\ 0 & 0 & \bar{Q}_{66} \end{bmatrix} \begin{Bmatrix} E_{11} \\ E_{22} \\ E_{12} \end{Bmatrix} \tag{3}$$

$$\begin{Bmatrix} m_{11} \\ m_{22} \\ m_{12} \\ m_{21} \end{Bmatrix} = \begin{bmatrix} B_{1111} & B_{1122} & 0 & 0 \\ B_{1122} & B_{1111} & 0 & 0 \\ 0 & 0 & B_{1212} & B_{1221} \\ 0 & 0 & B_{1221} & B_{1212} \end{bmatrix} \begin{Bmatrix} \Phi_{1,1} \\ \Phi_{2,2} \\ \Phi_{1,2} \\ \Phi_{2,1} \end{Bmatrix} \tag{4}$$

$\bar{Q}_{ij}$ are components of traditional in-plane stiffness matrix for the RVE(a thin plate element).

Since in the $X_1 - X_2$ plane, the properties of the material follows the isotropic relation, even for the couple stress relation, it can be shown that material constant B_{1122} does not enter into the formulation[10], and for the thin plate element considered, the displacement field can be specialized as

$$U_1 = U_{10} - X_3 \frac{\partial U_3}{\partial X_1}, U_2 = U_{20} - X_3 \frac{\partial U_3}{\partial X_2}, U_3 = U_{30}(X_1, X_2) \tag{5}$$

This displacement field leads to $\Phi_{1,1} = \dfrac{\partial^2 U_{30}}{\partial X_1 \partial X_2} = -\Phi_{2,2}$, so for the thin plate element, the couple stress relation can be further simplified. In order to keep the notation as for the classical thin plate theory, we note $M_{11} = m_{12}, M_{22} = -m_{21}, M_{12} = m_{22} = -m_{11}$, and $K_{11} = \Phi_{2,1}, K_{22} = \Phi_{1,2}, K_{12} = -\Phi_{1,1} = \Phi_{2,2}$, so the relation(4) can be arranged as

$$\begin{Bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} = \begin{bmatrix} \bar{\gamma} & -\bar{\beta} & 0 \\ -\bar{\beta} & \bar{\gamma} & 0 \\ 0 & 0 & \bar{\beta}+\bar{\gamma} \end{bmatrix} \begin{Bmatrix} K_{11} \\ K_{22} \\ K_{12} \end{Bmatrix} \tag{6}$$

$\bar{\beta}, \bar{\gamma}$ are the material constants for the couple stress applied to the case of a thin plate element with transverse symmetry X_3. Usually this non-classical materials constants are difficult to determine experimentally. The constitutive equations (6), (3) are applied for a material point (with special constraint condition such as thin plate element to simplify the analyses). From the micro-mechanical point of view, the stress and strain, couple stress and curvature are in fact the average value from the corresponding local quantities over the RVE. As arranged in the form presented in equations (3,6), it can be seen that $\bar{\beta}, \bar{\gamma}$ are in fact corresponding to the bending stiffness matrix for the thin plate element.

Connection with microstructures

From the classical thin plate theory and the centro-symmetry assumption, the relations between the average stress, strain, bending moment and curvature over the thin plate element are in this case

$$\begin{Bmatrix} <\sigma_{11}> \\ <\sigma_{22}> \\ <\sigma_{12}> \end{Bmatrix} = \begin{bmatrix} <Q_{11}> & <Q_{12}> & 0 \\ <Q_{12}> & <Q_{22}> & 0 \\ 0 & 0 & <Q_{66}> \end{bmatrix} \begin{Bmatrix} <\varepsilon_{11}> \\ <\varepsilon_{22}> \\ <\varepsilon_{12}> \end{Bmatrix} \tag{7}$$

$$\begin{Bmatrix} <M_{11}> \\ <M_{22}> \\ <M_{12}> \end{Bmatrix} = \begin{bmatrix} <x_3^2 Q_{11}> & <x_3^2 Q_{12}> & 0 \\ <x_3^2 Q_{12}> & <x_3^2 Q_{22}> & 0 \\ 0 & 0 & <x_3^2 Q_{66}> \end{bmatrix} \begin{Bmatrix} <k_{11}> \\ <k_{22}> \\ <k_{12}> \end{Bmatrix} \tag{8}$$

$<\bullet>$ means the volume average over the thin plate element. The centro-symmetry assures $<x_3 Q_{ij}>=0$, M_{ij} are the bending moment and k_{ij} are the corresponding curvature. It can be seen that for such thin plate element, the bending moment and the curvature are constant through the RVE due to the thin plate assumption.

In view of the similarity of equations (6),(8) and (3),(7), we see that for the couple stress theory applied to the thin plate element, the stress and strain relation can be estimated with the classical homogeneous procedure, and the couple stress and rotation relation can be evaluated with bending moment and curvature relation averaged over the RVE. This leads to the following expressions for non-classical material constants

$$\bar{\beta} = -\langle x_3^2 Q_{12} \rangle \qquad \bar{\gamma} = \langle x_3^2 Q_{11} \rangle \tag{9}$$

For homogeneous materials, it is found that these material constants are related to the element size *h*.

$$\bar{\beta} = -\frac{h^2}{3} Q_{12} \qquad \bar{\gamma} = \frac{h^2}{3} Q_{11} \tag{10}$$

where h is the half-thickness for the thin plate element.

For a layered composite plate, the local length scale such as the thickness of the each individual layer may enter into the constitutive formulation in the bending stiffness matrix. Consider a RVE consisting of two layered materials equally separated (Fig.1, due to the symmetry, only half of the element is shown). The Q_{ij}^1, Q_{ij}^2 are the in-plane stiffness tensors and t_1, t_2 are the corresponding thickness of each layers, and volume fraction of the two materials are f_1, f_2. From the basic thin plate theory, we get

$$< Q_{ij} >= f_1 Q_{ij}^1 + f_2 Q_{ij}^2 \tag{11}$$

$$< x_3^2 Q_{ij} >= \frac{1}{3}[Q_{ij}^1 f_1 + Q_{ij}^2 f_2]h^2 - \frac{1}{3}[Q_{ij}^1 f_1 t_2 + Q_{ij}^2 f_2 t_1]h - \frac{1}{6}[Q_{ij}^1 f_1 + Q_{ij}^2 f_2]t_1 t_2 \tag{12}$$

It is easy to check that when $Q_{ij}^2 = Q_{ij}^1$, equation (12) yields results for the homogeneous case. Equation (12) also leads to the following results

$$\frac{\bar{\beta}}{\beta_h} = 1 - \frac{1}{h}\frac{(f_1 Q_{12}^1 t_2 - f_2 Q_{12}^2 t_1)}{(f_1 Q_{12}^1 + f_2 Q_{12}^2)} - \frac{1}{2}\frac{t_1 t_2}{h^2}, \quad \frac{\bar{\gamma}}{\gamma_h} = 1 - \frac{1}{h}\frac{(f_1 Q_{11}^1 t_2 - f_2 Q_{11}^2 t_1)}{(f_1 Q_{11}^1 + f_2 Q_{11}^2)} - \frac{1}{2}\frac{t_1 t_2}{h^2} \tag{13}$$

where $\beta_h = \frac{1}{3}(f_1 Q_{12}^1 + f_2 Q_{12}^2)h^2, \gamma_h = \frac{1}{3}(f_1 Q_{11}^1 + f_2 Q_{11}^2)h^2$, corresponding to the homogeneous case with effective in-plane stiffness equal to $f_1 Q_{ij}^1 + f_2 Q_{ij}^2$.

It is seen that $< Q_{ij} >$ is just the average of stiffness matrix weighted by their volume fractions, and it corresponds to $\overline{Q}_{ij}$ in the couple stress theory. For this relation, the microstructure size effects does not appears. However for the coefficients of the bending stiffness matrix $< x_3^2 Q_{ij} >$, which corresponds to the coefficient matrix for the couple and rotation in the couple stress theory, the micro-structural parameters such as t_1, t_2 and the representative volume element size h enter into the constitutive formulation , as shown in equation (12).

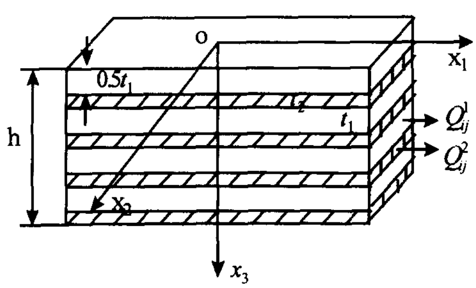

Fig.1 RVE with layered microstructures

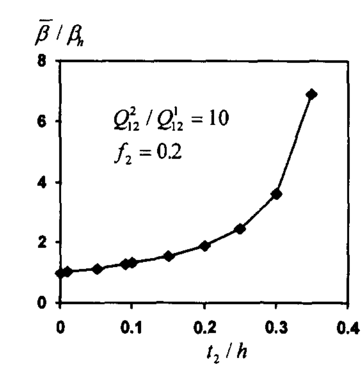

Fig 2 coefficient $\bar{\beta}/\beta_h$ for couple stress theory as function of micro-structural size

Fig 2 shows that influence of the number of the layers on the couple stress coefficients while the volume fractions f_1, f_2 are fixed.

It is seen that when the micro-structural sizes are comparable to the RVE size, the micro-structural size effects are significant. So in this situation, besides the relation between the stress and strain, another independent materials constants are necessary to relate the couple stress to the curvature. These materials constants are, as can be seen, related to the micro-structural size and the local material properties. When micro-structural sizes are fine enough compared to the h, equation (12) becomes

$$< x_3^2 Q_{ij} > = [f_1 Q_{ij}^1 + f_2 Q_{ij}^2]\frac{h^2}{3} = < Q_{ij} > \frac{h^2}{3} \qquad (14)$$

The thin layered plate element can be considered as a homogeneous material with the in-plane stiffness matrix given by $< Q_{ij} >$, this time the micro-structural size effect disappears in accordance with the classical micromechanical concept.

Discussion and Conclusions

It is shown, from the above analyses (based on a thin plate element), even for the homogeneous material, when the couple stress and curvature are considered, a length scale parameter appears, which is equal to the RVE size. When the RVE has microstructures (layered structure considered in this paper), a complex length parameter appears, depending on the microstructure size and their material properties. If the microstructure sizes are small enough compared to RVE size, the microstructure size effect can be neglected and the RVE can be considered as a homogeneous material.

Although in this paper a full connection between the couple stress theory and their implied microstructures is not given, a simple idea was indeed advanced to interpret the material constants from the microstructural parameters (geometry and material properties). It seems that the traditional RVE must be extended to include gradient effect in order to include the size effect into the constitutive formulation[11]. For a thin plate element, the obtained constitutive relations (stress-strain, moment-curvature) can be fit into macroscopic couple stress formulation, but the more general connection between the micro and macro is still being pursued

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