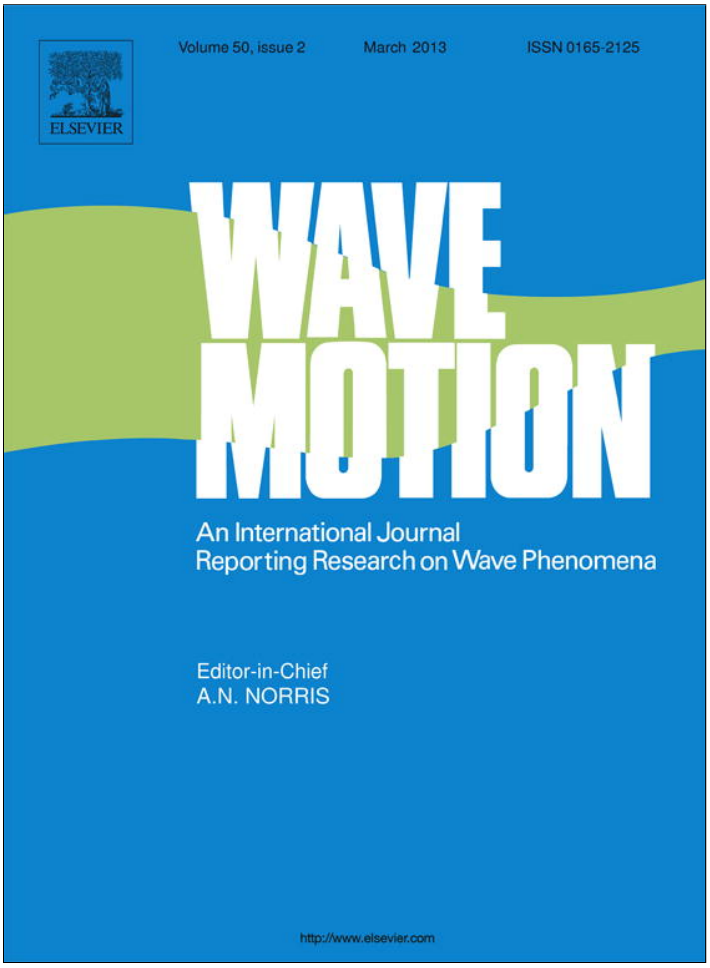
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Constraint condition on transformation relation for generalized acoustics

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ABSTRACT

For transformation acoustics (TA), the transformation relations for material and physical field are not unique when they are mapped from a virtual space to a physical space; the underlying mechanism is explored in this paper. We propose that the invariance of a physical process during a spatial mapping will impose the constraint condition for the transformation relation. This, together with the condition of energy conservation, provides a general method to derive the corresponding transformation relation for any physical process with the assumption of local affine transformation. When applied to TA, we show that the constraint conditions are not enough to determine the transformation relations completely, leaving a possibility to define them differently as found in the literature. New acoustic transformation relations with constant density or modulus are also proposed and validated numerically by constructing a two-dimensional acoustic cloak.

1. Introduction

For electromagnetic waves, the transformation method establishes the equivalency between a curved space and material space [1–4]. It provides an efficient way for finding material spatial distribution when a wave path is prescribed. With the development of electromagnetic metamaterials, many interesting devices have been proposed, including cloaks [3,5], concentrator and rotator [6,7], beam shifter/bender [8,9], and devices for illusion optics [10]. In parallel, with the help of acoustic metamaterials [11–13], acoustic devices have also been designed by transformation acoustics [14–19], including generalized acoustics, which consider acoustic waves in a more complex media in addition to classical fluid. Recently, an acoustic cloak has been demonstrated experimentally [20]. The transformation methods for electromagnetic wave and acoustic wave are called transformation optics (TO) and transformation acoustics (TA), respectively, both are the results of form-invariance of governing equations under an arbitrary coordinate transformation. However, unlike TO, which has a clear and unique relation between transformed and initial physical quantity (namely, permittivity, permeability, and electromagnetic fields) [21,22], there exist many different relations between the transformed and initial mass density and bulk modulus, as well as displacement and pressure for TA. For example, Chen and Chan [14] implicitly assumed that the pressure is unchanged during the transformation, and they derived the corresponding transformation relations for the mass density and bulk modulus. Milton et al. [15] supposed that the displacement has a special transformation, and consequently they derived the transformation relations for the other physical quantities for generalized acoustics. Their results differ from those proposed in [14]. Cummer et al. [16] proposed a new transformation relation for the displacement (or velocity), and stated that the transformation relation for the displacement proposed in [15] is not suitable for TA. Norris [17] showed that for a given mapping, the transformation relation for TA is not uniquely defined, and he pointed out that the transformation

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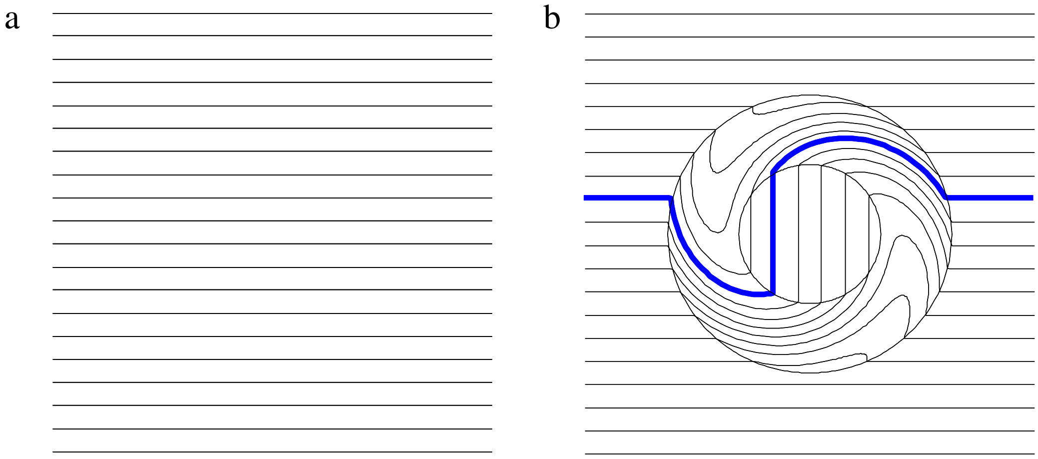

Fig. 1. A sketch of transformation method: (a) Initial space with a simple known field distribution; (b) deformed space characterizing the designed function.

relation for displacement proposed in [15] is possible. Therefore, different groups have come up with different formulations and there is no coherent theory that can link them together. Thus further efforts are still required to elucidate this ambiguity in TA, which is the objective of this study. We will examine the transformation method in a more general context, and figure out the constraint condition on the transformed field and material property for a given spatial mapping.

The paper is arranged as follows. In Section 2, the kinetics of a transformed variable during a mapping is assumed to follow the deformation induced by the spatial mapping; the invariance of a physical process provides constraint conditions necessary for deriving the transformation relation. In Section 3, application of these constraint conditions to TA is presented. We show that the constraint conditions are not sufficient to determine completely the transformation relations, therefore alternative ones can be proposed. In Section 4, discussions on the transformation relations of TO are provided, followed by conclusions.

2. Constraint condition imposed by the transformation method

2.1. Interpretation of transformation method

Consider a specific physical process prescribed in an initial space Ω, which is governed by a system of differential equation F, written in a Cartesian frame as

$$F[\mathbf{x}, t, \mathbf{C}(\mathbf{x}), \mathbf{u}(\mathbf{x}, t)] = 0, \quad \mathbf{x} \in \Omega, \tag{1}$$

where $\mathbf{x}$ is the spatial coordinate and t is the time; $\mathbf{C}$ and $\mathbf{u}$ represent the material property and related physical field, respectively. They are assumed to be continuous and have continuous derivatives within Ω [23]. The physical phenomenon described by Eq. (1) gives the relationship between $\mathbf{C}$ and $\mathbf{u}$ in every point within Ω. For a concrete physical problem, F, $\mathbf{C}$ and $\mathbf{u}$ will be specified together with initial and boundary conditions. In this section, we will keep our discussion as general as possible. Suppose there is a spatial mapping under which each point $\mathbf{x}$ in the region Ω is mapped to a point $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$ in a different space Ω', and F will retain its form, namely,

$$F[\mathbf{x}', t, \mathbf{C}'(\mathbf{x}'), \mathbf{u}'(\mathbf{x}', t)] = 0, \quad \mathbf{x}' \in \Omega', \tag{2}$$

then, the attached field $\mathbf{u}(\mathbf{x}, t)$ and material $\mathbf{C}(\mathbf{x})$ in Ω can also be respectively point-to-point mapped to Ω' as

$$\mathbf{C}'(\mathbf{x}') = T_C[\mathbf{C}(\mathbf{x})], \qquad \mathbf{u}'(\mathbf{x}', t) = T_u[\mathbf{u}(\mathbf{x}, t)]. \tag{3}$$

The new space Ω' is called the deformed space from continuum mechanics. Usually $\mathbf{C}(\mathbf{x})$ is set to be homogeneous and isotropic, the field $\mathbf{u}(\mathbf{x}, t)$ has a simple known distribution in Ω. One can choose carefully the mapping $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$ such that the mapped field $\mathbf{u}'(\mathbf{x}', t)$ in the deformed space follows a desired way, as illustrated schematically by a rotator in Fig. 1. Once the function is specified by $\mathbf{u}'(\mathbf{x}', t)$, according to Eq. (2) the related material to realize this function should be the transformed material $\mathbf{C}'(\mathbf{x}')$, which usually becomes inhomogeneous and anisotropic. According to this procedure, the transformation method provides a visualized way to redistribute the physical field point by point, and avoids inversely solving the governing equation to obtain the corresponding material distribution. Obviously, the transformation relations, Eq. (3), are essential to fulfill this method.

In order to obtain Eq. (3), one usually utilizes the methods based on the mathematical interpretation of the form-invariance between Eqs. (1) and (2). In this interpretation, the form-invariance is a pure mathematical property of Eq. (1) or (2). One needs first to write down the governing equation in a general curvilinear coordinate system to verify whether it has the same form as the original one. To this end, usually certain transformation relations for some physical quantities between $\mathbf{x}$ and $\mathbf{x}'$ have to be *pre-assumed*, and then the transformation relations for the other quantities are derived by the

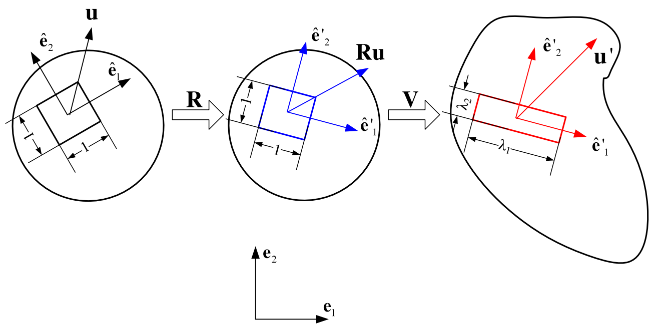

Fig. 2. Rigid rotation and stretch operation during a transformation at each point.

mathematical interpretation of the form-invariance. However, these *pre-assumed* transformation relations are not objective, and the resulted transformation relations are closely dependent on the *pre-assumed* ones.

To circumvent this difficulty, an alternative method will be proposed. Instead of finding the required transformation relation directly by verifying the invariance of governing equation, here we assume that the transformed governing equation is known and has the same form as the original one. According to the previous discussion, the same form of Eqs. (1) and (2) implies the invariance of the physical process during the mapping. This invariance in fact imposes the constraint on the transformed field $\mathbf{u}'(\mathbf{x}', t)$ and material $\mathbf{C}'(\mathbf{x}')$. If the kinetics of a transformed variable is prescribed during the mapping, together with Eq. (2) we can derive the transformation relation without any *pre-assumed* transformation. In the following, we will specify the kinetics for the transformed field and material by assuming that they will follow the deformation during the spatial mapping.

During the transformation, the material property and physical field related by Eq. (1) in the initial space are transported to the deformed space. The transformed material and field have to be subjected to some constraints, so that they should rebuild the same physical mechanism in the deformed space, i.e., satisfying Eq. (2). If we can establish a local Cartesian frame at each point in the deformed space, which is uniquely determined by the mapping, the governing equation written in the local Cartesian frame is form-invariant if we interpret a general mapping locally by an affine transformation point-by-point [24].

2.2. Local Cartesian frames

Consider a mapping $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$, which maps every point $\mathbf{x}$ in Ω unique to $\mathbf{x}'$ in a new space Ω', as shown in Fig. 2. It is useful to interpret $\mathbf{x}'$ as another Cartesian coordinate superposed on $\mathbf{x}$ [15]; if the mapping is interpreted by successive local affine transformation, then the mapping defines a deformation field on the initial space Ω, characterized by a deformation gradient tensor $\mathbf{A}$ with elements $A_{ij} = \partial x'_i/\partial x_j$. We ignore the translation part in the affine transformation because it does not affect the physical quantities. The tensor $\mathbf{A}$ can be further decomposed uniquely into a positive definite symmetric tensor and an orthogonal tensor [25]:

$$\mathbf{A} = \mathbf{V}\mathbf{R} = \mathbf{R}\mathbf{U}, \tag{4}$$

where $\mathbf{R}$ is an orthogonal tensor characterizing the rigid rotation of a point during the transformation, and $\mathbf{V}$ and $\mathbf{U}$ are the positive definite symmetric tensors describing pure stretch operations. We separately define λ_i and $\hat{\mathbf{e}}'_i$ as the eigenvalues and the corresponding eigenvectors of $\mathbf{V}$, i.e., $\mathbf{V} = \lambda_1\hat{\mathbf{e}}'_1\hat{\mathbf{e}}'_1 + \lambda_2\hat{\mathbf{e}}'_2\hat{\mathbf{e}}'_2 + \lambda_3\hat{\mathbf{e}}'_3\hat{\mathbf{e}}'_3$; thus, the eigenvectors $\hat{\mathbf{e}}'_i$ form a local Cartesian frame at each point in the deformed space Ω'. We also define $\hat{\mathbf{e}}_i$ by $\hat{\mathbf{e}}'_i = \mathbf{R}\hat{\mathbf{e}}_i$ (in fact, $\hat{\mathbf{e}}_i$ is the eigenvector of $\mathbf{U}$), and $\hat{\mathbf{e}}_i$ also forms a local Cartesian frame in the initial space Ω. $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}'_i$ can be different from the corresponding global frame $\mathbf{e}_i$, as illustrated in Fig. 2. By establishing these two local Cartesian frames $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}'_i$ for the initial and deformed spaces, respectively, we can write down the governing equation and transformation relation before and after the transformation in $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}'_i$, respectively.

Before proceeding further, we will discuss some properties of the established local Cartesian frames. During a mapping, an infinitesimal element $d\Omega$ (with the frame $\hat{\mathbf{e}}_i$ attached) will first be rotated with $\mathbf{R}$, and then rescaled by a factor λ_i in the $\hat{\mathbf{e}}'_i$ direction, and finally transformed to $d\Omega'$ (as illustrated in Fig. 2). Therefore, during the mapping, any physical quantity attached with the element will experience a rigid rotation operation $\mathbf{R}$ that rotates the physical quantity and the attached base $\hat{\mathbf{e}}_i$ in the initial space to $\hat{\mathbf{e}}'_i$ in the deformed space; then, a stretch operation $\mathbf{V}$ rescales the physical quantity accordingly in $\hat{\mathbf{e}}'_i$, as shown in Fig. 2. The rescaling should ensure the physical quantity to rebuild the same physical mechanism as that in the undeformed space.

To establish the differential relation between these two local Cartesian frames, consider a line element in $d\Omega$, $d\mathbf{x} = dx_i\mathbf{e}_i = d\hat{x}_i\hat{\mathbf{e}}_i$; during the mapping (local affine transformation), it is transformed to $d\mathbf{x}'$ in $d\Omega'$ as

$$d\mathbf{x}' = \mathbf{V}\mathbf{R}d\mathbf{x} = \lambda_i d\hat{x}_i\hat{\mathbf{e}}'_i = d\hat{x}'_i\hat{\mathbf{e}}'_i, \tag{5}$$

leading to the following differential operation between $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}'_i$:

$$\frac{\partial}{\partial \hat{x}'_i} = \frac{\partial}{\lambda_i \partial \hat{x}_i}. \tag{6}$$

2.3. Geometrical constraints

For the given mapping $\mathbf{x}' = \mathbf{x}'(\mathbf{x})$, we will determine the transformation relations for $(\hat{\mathbf{u}}', \hat{\mathbf{u}})$ as well as $(\hat{\mathbf{C}}', \hat{\mathbf{C}})$. In the initial space, the material properties are assumed to be isotropic, the governing equation is insensitive to the frame direction and can be written in $\hat{\mathbf{e}}_i$ as

$$F[\hat{\mathbf{x}}, t, \hat{\mathbf{C}}(\hat{\mathbf{x}}), \hat{\mathbf{u}}(\hat{\mathbf{x}}, t)] = 0, \quad \text{in } d\Omega. \tag{7}$$

As we know that $\hat{\mathbf{C}}$ and $\hat{\mathbf{u}}$ will first experience a rigid rotation $\mathbf{R}$ and then pure stretch operation along the eigenvectors of $\mathbf{V}$ to reach $\hat{\mathbf{C}}'$ and $\hat{\mathbf{u}}'$, we can symbolically write

$$\mathbf{V_C R} : \hat{\mathbf{C}} \mapsto \hat{\mathbf{C}}', \qquad \mathbf{V_u R} : \hat{\mathbf{u}} \mapsto \hat{\mathbf{u}}'. \tag{8}$$

In the established frame $\hat{\mathbf{e}}'_i$, $\mathbf{V_C}$, $\mathbf{V_u}$ have diagonal forms, which will be determined by the form-invariance of the governing equation during the transformation of $d\Omega$ to $d\Omega'$, because locally, both $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}'_i$ are Cartesian frames, and hence, we have

$$F[\hat{\mathbf{x}}', t, \hat{\mathbf{C}}'(\hat{\mathbf{x}}'), \hat{\mathbf{u}}'(\hat{\mathbf{x}}', t)] = 0 \quad \text{in } d\Omega'. \tag{9}$$

With the help of Eqs. (8) and (6), we can express Eq. (9) from the frame $\hat{\mathbf{e}}'_i$ to the frame $\hat{\mathbf{e}}_i$, and compare directly with Eq. (7) to determine $\mathbf{V_C}$, $\mathbf{V_u}$.

2.4. Energy conservation constraints

Since the mapping just transports a physical mechanism from the initial space to the deformed space, no new physical process manifests during the transformation. Therefore we can assume that at each element, there is no creation or loss of energy during the transformation; in addition, there is no interchange between the different types of energy. If the energy density is denoted by $w = w(\mathbf{C}, \mathbf{u})$ in the initial space, then the energy conservation leads to $w d\Omega = w' d\Omega'$, where w' is the energy density in the deformed space. With the help of the relation $d\Omega' = \lambda_1\lambda_2\lambda_3 d\Omega$, we have

$$w(\hat{\mathbf{C}}, \hat{\mathbf{u}}) = w'(\hat{\mathbf{C}}', \hat{\mathbf{u}}')\lambda_1\lambda_2\lambda_3. \tag{10}$$

The energy conservation will provide other constraint condition for $\mathbf{V_C}$ and $\mathbf{V_u}$. In the following, we will apply this general concept to acoustics.

3. Application to generalized acoustics

3.1. Constraint conditions for generalized acoustics

We consider a generalized acoustic wave equation in the context of pentamode materials (PMs) [17,26],

$$\begin{aligned} &\nabla \cdot \boldsymbol{\sigma} = \boldsymbol{\rho} \cdot \ddot{\mathbf{u}}, \\ &\boldsymbol{\sigma} = \kappa \, \mathrm{tr}(\mathbf{S}\nabla\mathbf{u})\mathbf{S}, \end{aligned} \tag{11}$$

where $\mathbf{S}$ is a general second-order tensor, $\mathbf{u}$ denotes the displacement vector, $\boldsymbol{\sigma}$ is the stress tensor, κ is the bulk modulus, and the density $\boldsymbol{\rho}$ is assumed to have a tensor form [27]. For PM, the corresponding material tensor $\mathbf{C} = \kappa \mathbf{S} \otimes \mathbf{S}$ can be realized at least theoretically [26]. When $\boldsymbol{\rho} = \rho\mathbf{I}$, $\mathbf{S} = \mathbf{I}$, and $\boldsymbol{\sigma} = p\mathbf{I}$, the following classical acoustic wave equation can be recovered:

$$\begin{aligned} &\nabla p = \rho \ddot{\mathbf{u}}, \\ &p = \kappa \nabla \cdot \mathbf{u}, \end{aligned} \tag{12}$$

where p is the acoustic pressure.

Now, we will write down the governing equation (Eq. (11)) in the local Cartesian frames $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{e}}'_i$, respectively. In the initial space, the acoustic wave equation is supposed to have a classical form given by Eq. (12). During the transformation, the material properties $\hat{\rho}\hat{\mathbf{I}}$, $\hat{\kappa}$ and the physical fields $\hat{p}\hat{\mathbf{I}}$, $\hat{\mathbf{u}}$ are transformed to $\hat{\boldsymbol{\rho}}'$, $\hat{\kappa}'$ and $\hat{\boldsymbol{\sigma}}'$, $\hat{\mathbf{u}}'$, respectively, and the tensor $\hat{\mathbf{S}} = \hat{\mathbf{I}}$ is transformed to $\hat{\mathbf{S}}'$. According to the previous analysis, there are two operations on each quantity during the transformation: first, a rotation operation from $\hat{\mathbf{e}}_i$ to $\hat{\mathbf{e}}'_i$ by $\mathbf{R}$, then a pure stretch operation in $\hat{\mathbf{e}}'_i$. Taking the displacement as an example, during the transformation, we have: $\hat{\mathbf{u}}' = \mathbf{V_u R}\hat{\mathbf{u}}$. As the frame $\hat{\mathbf{e}}'_i$ is specially established, which is the principle frame of the stretch $\mathbf{V}$, the pure stretch operation $\mathbf{V_u}$ on the displacement has a diagonal form in this principle frame, and we note it in $\hat{\mathbf{e}}_i'$ by $\mathbf{V_u} = \mathrm{diag}[f_1, f_2, f_3]$. For the rigid rotation operation, the vector $\hat{\mathbf{u}}$, together with the frame $\hat{\mathbf{e}}_i$, are rotated to the new local frame $\hat{\mathbf{e}}_i'$. Therefore, in the local frame $\hat{\mathbf{e}}_i'$, we still have $\mathbf{R}\hat{\mathbf{u}} = [\hat{u}_1, \hat{u}_2, \hat{u}_3]^{\mathrm{T}}$. Finally, the transformed displacement in the frame $\hat{\mathbf{e}}_i'$ can be written as $\hat{\mathbf{u}}' = [f_1\hat{u}_1, f_2\hat{u}_2, f_3\hat{u}_3]^{\mathrm{T}}$. The same idea is applied for the other quantities, and finally, we have

$$\begin{aligned}
\hat{\mathbf{S}}' &= \mathrm{diag}[d_1, d_2, d_3],\\
\hat{\boldsymbol{\sigma}}' &= \hat{p}\,\mathrm{diag}[e_1, e_2, e_3],\\
\hat{\mathbf{u}}' &= [f_1\hat{u}_1, f_2\hat{u}_2, f_3\hat{u}_3]^{\mathrm{T}},\\
\hat{\boldsymbol{\rho}}' &= \hat{\rho}\,\mathrm{diag}[g_1, g_2, g_3],\\
\hat{\kappa}' &= \hat{\kappa} h,
\end{aligned} \tag{13}$$

where d_i, e_i, f_i, g_i, and h are the scaling factors on the material property and physical field; they are constant on each transformed element as a result of the local affine transformation, and will be determined by the form-invariance of the governing equation. To this end, we write Eq. (12) in $\hat{\mathbf{e}}_i$ as

$$\begin{aligned}
&\frac{\partial \hat{p}}{\partial \hat{x}_i} = \hat{\rho}\ddot{\hat{u}}_i,\\
&\hat{p} = \hat{\kappa}\left(\frac{\partial \hat{u}_1}{\partial \hat{x}_1} + \frac{\partial \hat{u}_2}{\partial \hat{x}_2} + \frac{\partial \hat{u}_3}{\partial \hat{x}_3}\right).
\end{aligned} \tag{14}$$

After the mapping, the form-invariance of Eq. (11) implies that

$$\begin{aligned}
&\nabla \cdot \hat{\boldsymbol{\sigma}}' = \hat{\boldsymbol{\rho}}'\ddot{\hat{\mathbf{u}}}',\\
&\hat{\boldsymbol{\sigma}}' = \hat{\kappa}'\mathrm{tr}\left(\hat{\mathbf{S}}'\nabla\hat{\mathbf{u}}'\right)\hat{\mathbf{S}}'.
\end{aligned} \tag{15}$$

As in frame $\hat{\mathbf{e}}_i'$, the second-order tensors have diagonal forms, Eq. (15) can be written in index form without summation as

$$\begin{aligned}
&\frac{\partial \hat{\sigma}_{ii}'}{\partial \hat{x}_i'} = \hat{\rho}_{ii}'\ddot{\hat{u}}_i'\\
&\hat{\sigma}_{ii}' = \hat{\kappa}'\left(\hat{S}_{11}'\frac{\partial \hat{u}_1'}{\partial \hat{x}_1'} + \hat{S}_{22}'\frac{\partial \hat{u}_2'}{\partial \hat{x}_2'} + \hat{S}_{33}'\frac{\partial \hat{u}_3'}{\partial \hat{x}_3'}\right)\hat{S}_{ii}'.
\end{aligned} \tag{16}$$

Using Eqs. (6) and (13), Eq. (16) can be further written as

$$\begin{aligned}
&\frac{e_i}{\lambda_i}\frac{\partial \hat{p}}{\partial \hat{x}_i} = g_i f_i \hat{\rho}\ddot{\hat{u}}_i\\
&e_i\hat{p} = h\hat{\kappa}\left(\frac{d_1 f_1}{\lambda_1}\frac{\partial \hat{u}_1}{\partial \hat{x}_1} + \frac{d_2 f_2}{\lambda_2}\frac{\partial \hat{u}_2}{\partial \hat{x}_2} + \frac{d_3 f_3}{\lambda_3}\frac{\partial \hat{u}_3}{\partial \hat{x}_3}\right)d_i.
\end{aligned} \tag{17}$$

To derive Eq. (17), the following property is used: e.g., $d\hat{u}_i' = f_i d\hat{u}_i$ is employed for the increment of displacement. This is a natural consequence of the local affine transformation, because the scaling factors are constants in any infinitesimal element. By comparing Eq. (17) directly with Eq. (14), the following constraint conditions can be derived:

$$\frac{d_1 f_1}{\lambda_1} = \frac{d_2 f_2}{\lambda_2} = \frac{d_3 f_3}{\lambda_3}, \tag{18a}$$

$$\frac{h d_i^2}{g_i} = \lambda_i^2. \tag{18b}$$

The conservations for strain potential energy and kinetic energy lead to

$$\sum_{i=1}^{3}\hat{\sigma}_{ii}'\frac{\partial \hat{u}_i'}{\partial \hat{x}_i'} = \hat{p}\sum_{i=1}^{3}\frac{e_i f_i}{\lambda_i}\frac{\partial \hat{u}_i}{\partial \hat{x}_i} = \frac{\hat{p}}{\lambda_1\lambda_2\lambda_3}\sum_{i=1}^{3}\frac{\partial \hat{u}_i}{\partial \hat{x}_i}, \tag{18c}$$

$$\sum_{i=1}^{3}\hat{\rho}_{ii}'\dot{\hat{u}}_i'^2 = \hat{\rho}\sum_{i=1}^{3}g_i f_i^2\dot{\hat{u}}_i^2 = \frac{\hat{\rho}}{\lambda_1\lambda_2\lambda_3}\sum_{i=1}^{3}\dot{\hat{u}}_i^2. \tag{18d}$$

These then complement the following two additional constraint conditions:

$$\begin{aligned}
&e_i f_i = \frac{\lambda_i}{\lambda_1\lambda_2\lambda_3},\\
&g_i f_i^2 = \frac{1}{\lambda_1\lambda_2\lambda_3}.
\end{aligned} \tag{18e}$$

Totally, we have 11 equations for the 13 unknown scaling variables d_i, e_i, f_i, g_i, and h, therefore there is no unique solution, and we have some degrees of freedom to choose the transformation relations differently. To illustrate this possibility, in the following, we will give some examples. It can also be noted that $d_i/e_i = \sqrt{\lambda_1\lambda_2\lambda_3/h}$ in Eq. (18) indicates that the pentamode material model is always kept during the transformation.

3.2. Acoustic transformation with a constant pressure

Let the pressure p be unchanged during the transformation, i.e., $\hat{\boldsymbol{\sigma}}' = \hat{p}\hat{\mathbf{I}}'$ and $\hat{\mathbf{S}}' = \hat{\mathbf{I}}'$; therefore, we set $e_i = 1$ and $d_i = 1$, and then, from Eq. (18), the following unique solution for the scaling factors are found

$$\begin{aligned} &f_i = \frac{1}{\lambda_j\lambda_k}, \qquad g_i = \frac{\lambda_j\lambda_k}{\lambda_i}, \qquad h = \lambda_1\lambda_2\lambda_3, \\ &i, j, k = 1, 2, 3; \quad i \neq j, \quad i \neq k, \quad j \neq k. \end{aligned} \tag{19a}$$

Thus, in the frame $\hat{\mathbf{e}}_i'$, the transformation relations for the material property and physical field can be expressed as

$$\begin{aligned} &\hat{p}' = \hat{p}, \\ &\hat{\mathbf{u}}' = \frac{\mathrm{diag}[\lambda_1, \lambda_2, \lambda_3]}{\lambda_1\lambda_2\lambda_3}[\hat{u}_1, \hat{u}_2, \hat{u}_2]^{\mathrm{T}}, \\ &\hat{\boldsymbol{\rho}}' = \hat{\rho}\,\mathrm{diag}\left[\frac{\lambda_2\lambda_3}{\lambda_1}, \frac{\lambda_1\lambda_3}{\lambda_2}, \frac{\lambda_2\lambda_1}{\lambda_3}\right], \\ &\hat{\kappa}' = \hat{\kappa}\lambda_1\lambda_2\lambda_3, \end{aligned} \tag{19b}$$

or written in a tensor form in a global system due to objectivity of a tensor as

$$\begin{aligned} &p' = p, \\ &\mathbf{u}' = \frac{\mathbf{VRu}}{\lambda_1\lambda_2\lambda_3} = \frac{\mathbf{Au}}{\det\mathbf{A}}, \\ &\boldsymbol{\rho}' = \rho\frac{\lambda_1\lambda_2\lambda_3}{\mathbf{V}^2} = \rho\frac{\det\mathbf{A}}{\mathbf{A}\mathbf{A}^{\mathrm{T}}}, \\ &\kappa' = \kappa\det\mathbf{A}. \end{aligned} \tag{19c}$$

These transformation relations for p', $\boldsymbol{\rho}'$, and κ' agree with those obtained in [14,18], and that of $\mathbf{u}'$ agrees with the recent result obtained in [16] in an orthogonal system.

3.3. Acoustic transformation with a constant displacement

We now let the displacement remain unstretched, i.e., $\hat{u}_i' = \hat{u}_i$ or $f_i = 1$, and set $\hat{\mathbf{S}}' = \mathbf{V}/\det\mathbf{A}$ as proposed in [17], i.e., $d_1 = 1/(\lambda_2\lambda_3)$, $d_2 = 1/(\lambda_3\lambda_1)$, $d_3 = 1/(\lambda_1\lambda_2)$; subsequently, from Eq. (18), we can get the unique solution as

$$\begin{aligned} &e_i = \frac{1}{\lambda_j\lambda_k}, \qquad g_i = \frac{1}{\lambda_1\lambda_2\lambda_3}, \qquad h = \lambda_1\lambda_2\lambda_3, \\ &i, j, k = 1, 2, 3; \quad i \neq j, \quad i \neq k, \quad j \neq k. \end{aligned} \tag{20a}$$

The corresponding transformation relations in the global frame are given by

$$\boldsymbol{\sigma}' = \frac{\mathbf{A}p}{\det\mathbf{A}}, \qquad \mathbf{u}' = \mathbf{Ru}, \qquad \rho' = \frac{\rho}{\det\mathbf{A}}, \qquad \kappa' = \kappa\det\mathbf{A}. \tag{20b}$$

As $\mathbf{C}' = \kappa'\mathbf{S}' \otimes \mathbf{S}'$, it can be noted that the transformation relations for the modulus and density given by Eqs. (19) and (20) can be derived from each other with the following condition: $[\kappa', \rho_1', \rho_2', \rho_3'] \leftrightarrow [1/\rho', 1/C_{1111}', 1/C_{2222}', 1/C_{3333}']$. This result agrees with that proposed by Norris [17]. He also noted that the transformation relations given by Eq. (20) could prevent the mass singularity for acoustic cloaks. However, it should be mentioned that in addition to the pressure, the transformation relation for the displacement is also different in these two cases.

3.4. Acoustic transformation proposed by Milton et al. [15]

Milton et al. [15] propose the following transformation relation for displacement: $\mathbf{u}' = (\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{u}$; in the case of elastodynamic wave, this condition implies $\hat{u}_i' = \frac{\hat{u}_i}{\lambda_i}$ or $f_i = \frac{1}{\lambda_i}$, and using $\hat{\mathbf{S}}' = \mathbf{V}^2/\det\mathbf{A}$ as proposed in [17], i.e.,

$d_1 = \lambda_1/(\lambda_2\lambda_3)$, $d_2 = \lambda_2/(\lambda_3\lambda_1)$, $d_3 = \lambda_3/(\lambda_1\lambda_2)$, Eq. (18) leads to the following unique solution:

$$\begin{aligned} &e_i = \frac{\lambda_i}{\lambda_j\lambda_k}, \qquad g_i = \frac{\lambda_i}{\lambda_j\lambda_k}, \qquad h = \lambda_1\lambda_2\lambda_3, \\ &i, j, k = 1, 2, 3; \quad i \neq j, \quad i \neq k, \quad j \neq k. \end{aligned} \tag{21a}$$

The corresponding transformation relations in the global frame are derived as

$$\boldsymbol{\sigma}'_i = p\frac{\mathbf{A}\mathbf{A}^{\mathrm{T}}}{\det\mathbf{A}}, \qquad \mathbf{u}' = (\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{u}, \qquad \boldsymbol{\rho}' = \rho\frac{\mathbf{A}\mathbf{A}^{\mathrm{T}}}{\det\mathbf{A}}, \qquad \kappa' = \kappa\det\mathbf{A}, \tag{21b}$$

and $\mathbf{C}' = \kappa'\mathbf{S}' \otimes \mathbf{S}' = \frac{\kappa\mathbf{V}^2\otimes\mathbf{V}^2}{\det\mathbf{A}}$. We noted that the density tensor $\boldsymbol{\rho}'$ and the elasticity tensor $\mathbf{C}'$ have the same transformation relations as those presented in [15]. These confirm that the transformation relations given by Milton et al. [15] are also admissible for a generalized acoustic transformation based on PM theory.

3.5. *Acoustic transformation with one constant material property*

As discussed in Section 3.1, for acoustic transformation, we have 11 constraint equations for determining 13 scaling variables; therefore, we can propose some new transformation relations. Let us assume that the density is kept constant during the transformation: $\hat{\rho}'(\mathbf{x}') = \hat{\rho}(\mathbf{x})$ or $g_i = 1$. Let us further assume that $\hat{\kappa}' = \hat{\kappa}\det\mathbf{A}$ or $h = \lambda_1\lambda_2\lambda_3$, as proposed in [17]; then, from Eq. (18), the following unique solution can be derived:

$$\begin{aligned} &d_i = \left(\frac{\lambda_i}{\lambda_j\lambda_k}\right)^{\frac{1}{2}}, \qquad e_i = \left(\frac{\lambda_i}{\lambda_j\lambda_k}\right)^{\frac{1}{2}}, \qquad f_i = \left(\frac{1}{\lambda_1\lambda_2\lambda_3}\right)^{\frac{1}{2}}, \\ &i, j, k = 1, 2, 3; \quad i \neq j, \quad i \neq k, \quad j \neq k. \end{aligned} \tag{22a}$$

The corresponding transformation relations in the global frame are given by

$$\begin{aligned} &\mathbf{S}' = \left(\frac{\mathbf{A}\mathbf{A}^{\mathrm{T}}}{\det\mathbf{A}}\right)^{\frac{1}{2}}, \qquad \mathbf{p}' = p\left(\frac{\mathbf{A}\mathbf{A}^{\mathrm{T}}}{\det\mathbf{A}}\right)^{\frac{1}{2}}, \\ &\mathbf{u}' = (\det\mathbf{A})^{-\frac{1}{2}}\mathbf{R}\mathbf{u}, \qquad \rho' = \rho, \qquad \kappa' = \kappa\det\mathbf{A}. \end{aligned} \tag{22b}$$

and $\mathbf{C}' = \kappa'\mathbf{S}' \otimes \mathbf{S}'$; thus, without summation of index, we have

$$C'_{iiii} = \lambda_i^2\kappa. \tag{23}$$

Similarly, let the bulk modulus remain unchanged, $\hat{\kappa}'(\mathbf{x}') = \hat{\kappa}(\mathbf{x})$ or $h_i = 1$ and $d_i = 1$; then, the following acoustic transformation with constant modulus can also be obtained:

$$e_i = \left(\frac{1}{\lambda_1\lambda_2\lambda_3}\right)^{\frac{1}{2}}, \qquad f_i = \left(\frac{\lambda_i}{\lambda_j\lambda_k}\right)^{\frac{1}{2}}, \qquad g_i = \frac{1}{\lambda_i^2}, \tag{24a}$$

or

$$\begin{aligned} &\mathbf{S}' = \mathbf{I}, \qquad \kappa' = \kappa, \qquad \boldsymbol{\sigma}' = p\left(\frac{1}{\det\mathbf{A}}\right)^{\frac{1}{2}}, \\ &\mathbf{u}' = \left(\frac{\mathbf{A}\mathbf{A}^{\mathrm{T}}}{\det\mathbf{A}}\right)^{\frac{1}{2}}\mathbf{R}\mathbf{u}, \qquad \boldsymbol{\rho}' = \rho\left(\frac{1}{\mathbf{A}\mathbf{A}^{\mathrm{T}}}\right). \end{aligned} \tag{24b}$$

The transformation relations for the modulus and density given by Eqs. (22) and (24) also have the following symmetry: $[1/\rho', 1/C'_{1111}, 1/C'_{2222}, 1/C'_{3333}] \leftrightarrow [\kappa', \rho'_1, \rho'_2, \rho'_3]$. We should point out that among the above-mentioned transformations, only the transformation relations given by Eqs. (19) and (24) keep the transformed medium as a fluid, and the other transformations convert a fluid to a more complex material, called the pentamode material.

To validate the proposed ρ-unchanged transformation, in the following, we construct a two-dimensional acoustic cloak with the transformation relations for modulus and density given by Eq. (22). For a cylindrical cloak, the ρ-unchanged transformation given by Eq. (22) requires the principal stretches $\lambda_r, \lambda_\theta, \lambda_z$ to be unity at the outer boundary, in order to satisfy the displacement and pressure continuity conditions [17]. Usually, the outer boundary is fixed during a spatial deformation in constructing cloaks, and hence, λ_θ and λ_z naturally become unity at the outer boundary; however, λ_r at the outer boundary depends on the transformation. The linear transformation $r' = a + r(b - a)/b$ is not applicable to this ρ-unchanged transformation, because $\lambda_r = dr'/dr = (b - a)/b \neq 1$ at the outer boundary, where a and b are the radii of the inner and outer boundary of the cloak, respectively. In the following, the nonlinear transformation $r' = ab^2/[(a - b)r + b^2]$

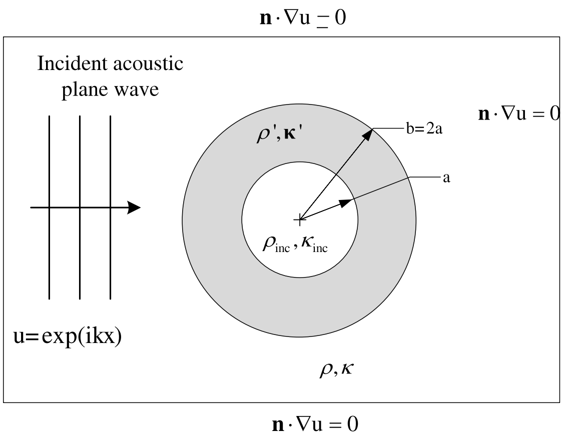

Fig. 3. Computational domain and simulation for a left-incident acoustic cloak based on Helmholtz equation.

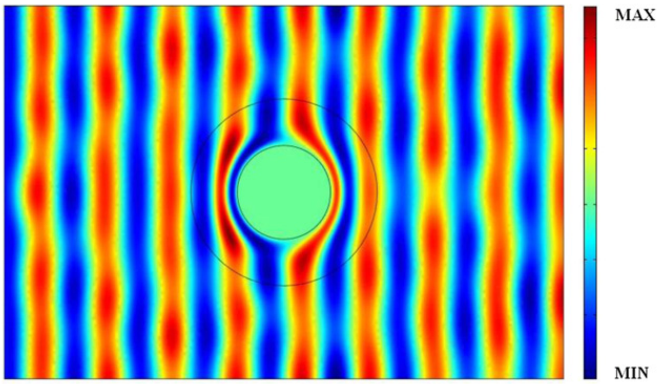

Fig. 4. Simulation of the displacement field around the ρ-unchanged acoustic cloak.

proposed in [28] will be used, and at the outer boundary, $\lambda_r(r = b) = (b - a)/a = 1$, if $b = 2a$. From this transformation, we can compute **A** everywhere in the cloak, and subsequently, the material distributions necessary for realizing this cloak are given by Eqs. (22) and (23).

To validate the proposed cloak, we consider a plane acoustic wave incident on the cloak. For a plane wave, the displacement in Eq. (12) can be expressed by a scalar u. Eliminating p in Eq. (12) gives the wave equation for the scalar displacement, i.e., the reduced acoustic equation $\nabla \cdot (\kappa \nabla u) - \rho \ddot{u} = 0$. Thus, the same PDE mode (Helmholtz equations) $\nabla \cdot (\mathbf{c} \nabla p) + ap = 0$ of commercial software COMSOL Multiphysics can be used to demonstrate the cloaking effect for the harmonic wave, where **c** is a tensor representing the elasticity tensor, just as the method used in [29]. Here, we set $c'_{ii} = \lambda_i^2 \kappa$ from Eq. (23) and $a' = a$. As COMSOL solver requires Cartesian coordinates, it is necessary to write **c** in the global Cartesian coordinate by the tensor transformation rule, i.e., $c'_{xx} = c'_{rr} \cos^2\theta + c'_{\theta\theta} \sin^2\theta$, $c'_{xy} = c'_{yx} = (c'_{rr} - c'_{\theta\theta}) \sin\theta \cos\theta$, and $c'_{yy} = c'_{rr} \sin^2\theta + c'_{\theta\theta} \cos^2\theta$. Fig. 3 shows the computational domain for a horizontally incident wave, and $a = 0.2$ m, $b = 0.4$ m. In the simulation, the background medium is set to be water, $\rho = 1 \times 10^3$, $\kappa = 2.18 \times 10^9$ in SI units, and the wavelength of the incident wave is 0.35 m. The material parameters within the cloak are $\rho_{\text{inc}} = \rho/5$ and $\kappa_{\text{inc}} = \kappa$. The simulation of the cloak constructed by the proposed ρ-unchanged transformation is shown in Fig. 4, and the simulation result confirms the validity of the proposed transformation. The imperfection of the simulated result is believed to come from the numerical simulation. When the Helmholtz equation is solved in the simulation, the parameter **c** will tend to infinity with a higher order than the stretch λ_θ near the inner boundary (see Eq. (23)), therefore more refined discretization is needed with a cost of computation time to obtain more perfect results.

4. Discussions and conclusions

The proposed method can be applied to other wave phenomena. Taking the electromagnetic wave as an example, Maxwell's equations in Cartesian coordinate read

$$\nabla \times \mathbf{E} = -\boldsymbol{\mu}\dot{\mathbf{H}}, \qquad \nabla \times \mathbf{H} = +\boldsymbol{\varepsilon}\dot{\mathbf{E}}. \tag{25}$$

Maxwell's equations possess a special symmetry, which indicates that the material parameters and fields should have the same transformation, respectively, or more explicitly, $\boldsymbol{\varepsilon}' = \boldsymbol{\mu}'$ and $\mathbf{E}' = \mathbf{H}'$, if $\boldsymbol{\varepsilon} = \boldsymbol{\mu}$ and $\mathbf{E} = \mathbf{H}$. This condition makes it possible to analyze the electromagnetic transformation only by one of the equations in Eq. (25). The material parameters in the initial space are assumed to be isotropic, i.e., $\boldsymbol{\varepsilon} = \varepsilon\mathbf{I}$ and $\boldsymbol{\mu} = \mu\mathbf{I}$. According to the method proposed in Section 2, the transformation relations for the material property and physical field take the following forms in the local Cartesian frame $\hat{\mathbf{e}}_i'$:

$$\begin{aligned}
\hat{\boldsymbol{\varepsilon}}' &= \hat{\varepsilon}\,\mathrm{diag}[a_1, \quad a_2, \quad a_3], \\
\hat{\boldsymbol{\mu}}' &= \hat{\mu}\,\mathrm{diag}[a_1, \quad a_2, \quad a_3], \\
\hat{\mathbf{E}}' &= [b_1\hat{E}_1 \quad b_2\hat{E}_2 \quad b_3\hat{E}_3]^{\mathrm{T}}, \\
\hat{\mathbf{H}}' &= [b_1\hat{H}_1 \quad b_2\hat{H}_2 \quad b_3\hat{H}_3]^{\mathrm{T}}.
\end{aligned} \tag{26}$$

It is easy to establish the following unique solution according to the method proposed in Section 2:

$$a_i = \frac{\lambda_i}{\lambda_j\lambda_k}, \qquad b_i = \frac{1}{\lambda_i}, \quad i, j, k = 1, 2, 3;\ i \neq j,\ i \neq k,\ j \neq k. \tag{27}$$

The corresponding transformation relations in the global frame are derived as

$$\boldsymbol{\varepsilon}' = \frac{\mathbf{A}\varepsilon\mathbf{A}^{\mathrm{T}}}{\det\mathbf{A}}, \qquad \boldsymbol{\mu}' = \frac{\mathbf{A}\mu\mathbf{A}^{\mathrm{T}}}{\det\mathbf{A}}, \qquad \mathbf{E}' = (\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{E}, \qquad \mathbf{H}' = (\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{H}. \tag{28}$$

Eq. (28) is the same as the known result in the literature. It is also shown that the transformation relations for electromagnetic transformation are uniquely determined.

For elastic waves, it is shown that the governing equation in the deformed space should have different form from the original one [30,31], thus the proposed local invariance assumption fails. However, the local affine transformation will lead to high frequency approximation [32,33], which is useful in elastic ray theory [34,35]. Norris and Shuvalov [36]and Vasquez et al. [37] also developed a comprehensive theory for elastic transformation based on the mathematical interpretation of form-invariance.

To conclude, we developed a general method to derive the transformation relations during a spatial mapping. The method is based on the physical interpretation of local form-invariance, and on the kinetics of the transformed field and materials and the energy conservation condition. No pre-assumed transformation relations are necessary. For acoustic wave, we derived the constraint condition and found that the constraint conditions are less than the scaling variables for TA, this provides a possibility to define transformation relations differently; it also explains the different acoustic transformations existing in the literature. New acoustic transformations with constant density or modulus are also proposed and validated by constructing a two-dimensional acoustic cloak.

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