Elastic wave transparency of a solid sphere coated with metamaterials

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The elastic wave transparency phenomenon of a solid sphere coated with metamaterials is investigated in a solid host medium having nonzero shear modulus. The first three scattering coefficients of the coated sphere are derived in the Rayleigh limit and expressed in terms of the effective parameters of the coated sphere assemblage. It is found that the effective bulk modulus, mass density, and shear modulus of the coated sphere system dominate the zeroth, first, and second order scattering effects, respectively. Quasistatic transparency conditions are obtained by setting these scattering coefficients to be zero. It is also shown that the obtained transparency conditions are the same as those derived from the neutral inclusion concept. Obtained results from full-wave analyses show that the given conditions can well predict the transparency induced by metamaterials even in the regime far beyond the Rayleigh limit.

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I. INTRODUCTION

Electromagnetic metamaterials that can have any property values in material space have led to a new research area in optics and electromagnetics.[1–3] The unusual properties of these materials offer a great opportunity in designing superior optical and microwave devices. Recently, it has been found that metamaterials have the unique cloaking effect, which can be employed to make an object transparent or invisible.[4–6] One approach to realize the transparency is based on the coordinate transformation method, with which electromagnetic wave can be guided around an object, as if the object is not there.[6–9] However, the metamaterial used for the cloak is highly anisotropic and usually has to be simplified for experimental realization.[10]

As a different approach, a small particle can be made electromagnetically transparent by coating it with an isotropic plasmonic metamaterial.[5] This can be further illustrated with the concept of neutral inclusion in the quasistatic limit.[11] The underlying physical mechanism is to induce an oppositely signed electric dipolar field within the cover to cancel the field produced by the object. Thus, the transparency is not sensitive to the imperfection of the objects to be cloaked.[12,13] This method is suitable to cloak objects with dimensions smaller than the operating wavelength, since multipole contributions from a relatively large object cannot be simply canceled by tuning the cloaking parameters. However, several transparent coated spheres joined together to form an object with large electrical size can still be transparent.[14] This may provide a new way to achieve transparency for an object with size larger than the wavelength.

Elastic media with specific microstructures can also exhibit anomalous overall properties in the resonate state,[15] such as negative bulk modulus,[16] negative mass density,[17,18] or both.[19,20] This type of materials can be broadly classfied as acoustic wave or elastic wave metamaterials. Analogous to the phenomenon of electromagnetic cloaking, a counterpart exists correspondingly for elastic waves. Contrary to the invariant form of the Maxwell equation, Milton *et al.*[21] stated that the equations of motion for elastodynamics are not transformation invariant. However, there is a transformation invariant form for two-dimensional acoustic equations, and a cylindrical metamaterial cloak with anisotropic mass densities has been validated by full-wave simulations.[22]

In a previous paper,[23] we studied the acoustic transparency realized with isotropic metamaterials in a fluid system having zero shear modulus. The neutral inclusion concept was used to derive the quasistatic transparency conditions for a multilayered sphere. However, due to the fluid nature of the material, experimental realization of the coated sphere becomes challenging. It is therefore of significant practical interest to examine whether a solid object can be cloaked with a solid metamaterial for an incident compressional wave and study how the excited P (longitudinal) and S (transverse) scattered waves are eliminated simultaneously with the cloak when the surrounding medium has nontrivial shear modulus. These issues will be addressed in the present paper.

II. THEORETICAL ANALYSIS

A. Scattering coefficients of a coated sphere in the Rayleigh limit

The configuration of concern is shown in Fig. 1, where a coated sphere is placed in a host material. Each constituent of the composite system is characterized by bulk modulus κ_i, shear modulus μ_i, and mass density ρ_i, with the subscript $i=1,2,3$ representing separately the sphere, the coating, and the host medium. Let r_1 denote the radius of the uncoated sphere and r_2 the radius of the coated sphere. A plane harmonic compressional wave propagates along the positive direction of the z axis. If the host is a solid material with a nontrivial shear modulus, the coated sphere will scatter not only P waves but also S waves due to the coupling mode effect. It is necessary to study in detail the scattering coefficients of the coated sphere, since the cloaking effect of the metamaterial cover depends on the far-field scattering property.[4] Exact solutions for the scattering fields of a coated

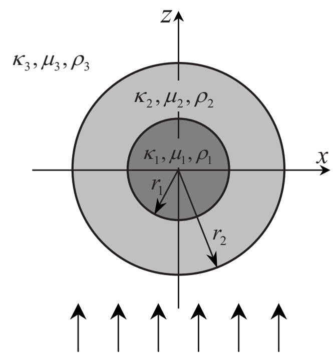

FIG. 1. A coated sphere embedded in a host medium and illuminated by a plane compressional wave.

sphere have already been obtained many years ago.[24] For completeness, a brief introduction is presented below for further discussions.

In the spherical coordinate system (r,θ,φ), the incident longitudinal wave is characterized by a displacement potential,

$$\Phi = \sum_{n=0}^{\infty} (2n+1) i^n j_n(\alpha_3 r) P_n(\cos\theta). \tag{1}$$

The scattered waves from the composite sphere are related to the displacement potentials Φ and Ψ, representing, respectively, the scattered P and S waves, which are expressed as

$$\Phi = -\sum_{n=0}^{\infty} a_n h_n(\alpha_3 r) P_n(\cos\theta), \tag{2}$$

$$\Psi = -\sum_{n=0}^{\infty} b_n h_n(\beta_3 r) P_n(\cos\theta), \tag{3}$$

where $j_n(z)$ is the spherical Bessel function, $h_n(z)$ is the spherical Hankel function of the first kind, $P_n(x)$ is the Legendre polynomial, and a_n and b_n are the unknown scattering coefficients of longitudinal and transverse waves. The propagation constants α_3 and β_3 are given by

$$\alpha_3 = \omega\sqrt{\frac{\rho_3}{\kappa_3 + 4\mu_3/3}}, \tag{4}$$

$$\beta_3 = \omega\sqrt{\frac{\rho_3}{\mu_3}}. \tag{5}$$

With the potentials Φ and Ψ determined, the displacements can be expressed as

$$\mathbf{u} = \nabla\Phi + \nabla\left(\mathbf{e}_\varphi \frac{\partial\Psi}{\partial\theta}\right). \tag{6}$$

Note that for incident compressional waves, $\Psi=0$. The stress components are related to the displacements as

$$\boldsymbol{\sigma} = (\kappa - 2\mu/3)(\nabla\cdot\mathbf{u})\mathbf{I} + \mu(\nabla\mathbf{u} + \mathbf{u}\nabla). \tag{7}$$

In the limiting case of $n=0$ order, $\partial\Psi/\partial\theta=0$ leads to $\mathbf{u}=\nabla\Phi$. Thus, b_0 is not important and can have any value. The zeroth order scattered wave is therefore a dilational wave.

The displacement and stress fields within the coated sphere can be defined in the same way by the corresponding displacement potentials. Six unknown scattering coefficients will be used to describe the P and S waves localized inside the composite sphere. At the sphere-coating interface $r=r_1$ and the external surface $r=r_2$, the normal and tangential components of displacements and stresses should be continuous, resulting in eight equations for eight unknowns for every order n except $n=0$. For $n=0$, not only the tangential components of the displacement and stress vanish but also there are no shear waves in the system, yielding four equations for four unknowns in this limiting case. The scattering coefficients a_n and b_n are then determined uniquely, which are used to define the total scattering cross section Q_{sca} of the coated sphere as

$$Q_{\mathrm{sca}} = w_3^2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)\pi}\left[|a_n|^2 + n(n+1)\frac{\alpha_3}{\beta_3}|b_n|^2\right], \tag{8}$$

where $w_3=2\pi/\alpha_3$ is the wavelength of the compressional wave in the host medium.

In the Rayleigh limit $(z \ll 1)$, where terms of order z^2 and higher for spherical Bessel and Hankel functions $j_n(z)$ and $h_n(z)$ appearing in the displacement potentials are negligible, only the first few scattering coefficients contribute to the final scattering. Under such conditions, analytical expressions of the scattering coefficients a_n and b_n can be derived. For convenience of discussion, the following parameters $\kappa_{\mathrm{eff}}^{\mathrm{HS}}$, $\mu_{\mathrm{eff}}^{\mathrm{HS}}$, $\rho_{\mathrm{eff}}^{\mathrm{M}}$, and $\rho_{\mathrm{eff}}^{\mathrm{B}}$ are introduced:

$$\kappa_{\mathrm{eff}}^{\mathrm{HS}}/\kappa_2 = 1 + \frac{f(\kappa_1-\kappa_2)}{\kappa_2 + (1-f)p(\kappa_1-\kappa_2)}, \tag{9}$$

with

$$p = \frac{3\kappa_2}{3\kappa_2 + 4\mu_2},$$

$$\mu_{\mathrm{eff}}^{\mathrm{HS}}/\mu_2 = 1 + \frac{f(\mu_1-\mu_2)}{\mu_2 + (1-f)q(\mu_1-\mu_2)}, \tag{10}$$

with

$$q = \frac{6}{5}\frac{\kappa_2 + 2\mu_2}{3\kappa_2 + 4\mu_2},$$

$$\rho_{\mathrm{eff}}^{\mathrm{M}}/\rho_2 = 1 + f\frac{\rho_1-\rho_2}{\rho_2}, \tag{11}$$

$$\rho_{\mathrm{eff}}^{\mathrm{B}}/\rho_2 = 1 + \frac{3f(\rho_1-\rho_2)}{3\rho_2 + 2(1-f)(\rho_1-\rho_2)}, \tag{12}$$

where $f=(r_1/r_2)^3$. The physical meaning of the above nondimensional material parameters will be discussed later. With these parameters, the first three scattering coefficients a_n and

b_n ($n \leqslant 2$) in the Rayleigh limit can be expressed in a concise form. For different configurations classified by the solid and fluid nature of materials, the scattering coefficients are presented below, for which those terms higher than order z^3 have been neglected.

Case I. Solid shell and solid host material.

$$a_0 = i\frac{\kappa_{\text{eff}}^{\text{HS}} - \kappa_3}{3\kappa_{\text{eff}}^{\text{HS}} + 4\mu_3}(\alpha_3 r_2)^3, \tag{13a}$$

$$a_1 = \frac{\rho_{\text{eff}}^{\text{M}} - \rho_3}{3\rho_3}(\alpha_3 r_2)^3, \tag{13b}$$

$$b_1 = -\frac{\rho_{\text{eff}}^{\text{M}} - \rho_3}{3\rho_3}\alpha_3\beta_3^2 r_2^3, \tag{13c}$$

$$a_2 = -\frac{20i\mu_3(\mu_{\text{eff}}^{\text{HS}} - \mu_3)/3}{6\mu_{\text{eff}}^{\text{HS}}(\kappa_3 + 2\mu_3) + \mu_3(9\kappa_3 + 8\mu_3)}(\alpha_3 r_2)^3, \tag{13d}$$

$$b_2 = \frac{10i\mu_3(\mu_{\text{eff}}^{\text{HS}} - \mu_3)/3}{6\mu_{\text{eff}}^{\text{HS}}(\kappa_3 + 2\mu_3) + \mu_3(9\kappa_3 + 8\mu_3)}(\beta_3 r_2)^3. \tag{13e}$$

Case II. Fluid shell and solid host material.

$$a_0 = i\frac{\kappa_{\text{eff}}^{\text{HS}} - \kappa_3}{3\kappa_{\text{eff}}^{\text{HS}} + 4\mu_3}(\alpha_3 r_2)^3, \tag{14a}$$

$$a_1 = \frac{\rho_{\text{eff}}^{\text{B}} - \rho_3}{3\rho_3}(\alpha_3 r_2)^3, \tag{14b}$$

$$b_1 = -\frac{\rho_{\text{eff}}^{\text{B}} - \rho_3}{3\rho_3}\alpha_3\beta_3^2 r_2^3, \tag{14c}$$

$$a_2 = \frac{20i\mu_3}{3(9\kappa_3 + 8\mu_3)}(\alpha_3 r_2)^3, \tag{14d}$$

$$b_2 = -\frac{10i\mu_3}{3(9\kappa_3 + 8\mu_3)}(\beta_3 r_2)^3. \tag{14e}$$

Case III. Solid shell and fluid host material.

$$a_0 = i\frac{\kappa_{\text{eff}}^{\text{HS}} - \kappa_3}{3\kappa_{\text{eff}}^{\text{HS}}}(\alpha_3 r_2)^3, \tag{15a}$$

$$a_1 = \frac{\rho_{\text{eff}}^{\text{M}} - \rho_3}{2\rho_{\text{eff}}^{\text{M}} + \rho_3}(\alpha_3 r_2)^3. \tag{15b}$$

Case IV. Fluid shell and fluid host material.

$$a_0 = i\frac{\kappa_{\text{eff}}^{\text{HS}} - \kappa_3}{3\kappa_{\text{eff}}^{\text{HS}}}(\alpha_3 r_2)^3, \tag{16a}$$

$$a_1 = \frac{\rho_{\text{eff}}^{\text{B}} - \rho_3}{2\rho_{\text{eff}}^{\text{B}} + \rho_3}(\alpha_3 r_2)^3. \tag{16b}$$

In every case, the parameter b_0 is not important and thus not given here.

B. Transparency and resonance conditions of a coated sphere

Equations (13)–(16) convey rich information about the cloaking effect of metamaterials. First of all, the physical meanings of the defined parameters $\kappa_{\text{eff}}^{\text{HS}}$, $\mu_{\text{eff}}^{\text{HS}}$, $\rho_{\text{eff}}^{\text{M}}$, and $\rho_{\text{eff}}^{\text{B}}$ are examined. For a composite filled with coated spheres that are randomly distributed in the host medium and have gradual sizes in order to fill the whole space, $\kappa_{\text{eff}}^{\text{HS}}$ and $\mu_{\text{eff}}^{\text{HS}}$ denote its effective bulk modulus and effective shear modulus calculated with the Hashin-Shtrikman (HS) bound,[25] $\rho_{\text{eff}}^{\text{M}}$ is the effective mass density obtained by the volume averaged method, whereas $\rho_{\text{eff}}^{\text{B}}$ is the effective mass density calculated with Berryman's formula.[26] With these effective parameters, Eqs. (13)–(16) represent also the solutions of a single sphere having material parameters $\kappa_{\text{eff}}^{\text{HS}}$, $\mu_{\text{eff}}^{\text{HS}}$, and $\rho_{\text{eff}}^{\text{M}}$ (or $\rho_{\text{eff}}^{\text{B}}$) embedded in a host material with properties κ_3, μ_3, and ρ_3.[27] This implies that in the Rayleigh limit, a coated sphere can be equivalent to a homogeneous effective sphere. The coated sphere and its effective sphere have almost the same scattering fields in the host medium, if contributions from higher order scattering are omitted.

It is well known that the HS bound model corresponds to a coated sphere assemblage system. The HS bound will give rigorous predictions of the effective bulk and shear moduli in the long wavelength limit. The effective mass density of a coated sphere assemblage obeys different rules depending on whether the cover is solid or fluid material. Rigorous derivation based on multiple scattering theory has revealed that the mixing rule is valid in the assumption of wave field homogeneity.[28] This assumption is often implicitly valid for the case of a solid shell. However, when the core and shell materials have large impedance mismatch, especially in the case of a fluid shell, the effective mass density follows the Berryman's formula, since the homogeneous field assumption has been violated.

Based on Eqs. (13)–(16), the transparency phenomenon of a coated sphere can be investigated mathematically by systematically reducing the scattering coefficients. If the effective parameters satisfy $\kappa_{\text{eff}}^{\text{HS}}=\kappa_3$, $\mu_{\text{eff}}^{\text{HS}}=\mu_3$, and $\rho_{\text{eff}}^{\text{M}}=\rho_3$ for a solid shell or $\rho_{\text{eff}}^{\text{B}}=\rho_3$ for a fluid shell, the scattering coefficients of the first three orders are zero in cases I, III, and IV. In the Rayleigh limit ($z \ll 1$), the total scattering cross section will be very small. In this case, an outside observer will hardly detect the coated sphere from the scattered waves it receives. For case II, a_2 and b_2 always involve terms of order z^3, which only relate to the parameters of the host material. In this case, there are no design parameters to choose from for the coated sphere and one has to let $\mu_3=0$ for a better transparency.

Physically, the above transparency conditions are equivalent to those obtained with the neutral inclusion concept. Since a coated sphere can be represented by its effective sphere in the Rayleigh approximation, the effective parameters of the coated sphere are the same as those of the surrounding medium. The scattering due to the coated sphere can be very small, and hence the observer will not "see" the sphere. This is exactly the physical meaning of the neutral inclusion concept, which has been discussed previously for the design of transparency.[11,23,29] When the shell (coating) is fluid, the solid nature of the core is concealed by the fluid shell so that the shear waves are localized in the inner sphere and shielded by the cover. In the host medium, the shear waves are excited again, independent of the material parameters of the coated sphere. Consequently, to achieve transparency, the shear modulus of the host material must be zero. With the concept of neutral inclusion, nonspherical coated objects can also be made transparent, as demonstrated for electromagnetic wave.[11]

A careful examination of Eqs. (13)–(16) reveals that the first three angular scattering channels are characterized by the effective bulk modulus, mass density, and shear modulus, respectively. In the Rayleigh limit, these channels are separate and parallel, and hence by tuning material parameters, the separate realizations of transparency in each channel can lead to the overall transparency. Consequently, the following general transparency conditions stand: $\kappa_{\text{eff}}=\kappa_3$, $\mu_{\text{eff}}=\mu_3$, and $\rho_{\text{eff}}=\rho_3$. With the help of Eqs. (9)–(12), the quasistatic transparency conditions of a coated sphere are given by

$$\frac{(\kappa_3-\kappa_2)[p\kappa_1+(1-p)\kappa_2]}{(\kappa_1-\kappa_2)[p\kappa_3+(1-p)\kappa_2]}=\frac{r_1^3}{r_2^3}, \tag{17}$$

$$\frac{(\mu_3-\mu_2)[q\mu_1+(1-q)\mu_2]}{(\mu_1-\mu_2)[q\mu_3+(1-q)\mu_2]}=\frac{r_1^3}{r_2^3}, \tag{18}$$

$$\frac{\rho_2-\rho_3}{\rho_2-\rho_1}=\frac{r_1^3}{r_2^3}, \tag{19a}$$

for a cover with nonzero shear modulus, and

$$\frac{(\rho_2-\rho_3)(\rho_2+2\rho_1)}{(\rho_2-\rho_1)(\rho_2+2\rho_3)}=\frac{r_1^3}{r_2^3}, \tag{19b}$$

for a cover with zero shear modulus, where the expressions of p and q have been given previously. For a solid host material, i.e., cases I and II, both the compressional and shear waves are scattered in the region exterior of the sphere and must be minimized simultaneously for the transparency. From Eqs. (13) and (14) as well as physical understanding based on the neutral inclusion concept, the conditions (17)–(19) always fulfill this requirement. In comparison, for a fluid host material, i.e., cases III and IV, only the compressional waves are scattered and the second order scattering coefficient is negligible. Equations (17) and (19) therefore suffice to determine the transparency.

Equations (13)–(16) can also be used to determine the conditions for the resonance phenomenon to occur. To this end, a coated sphere will exhibit an extremely large total scattering cross section. Mathematically, this effect is induced by the vanishing denominator of the scattering coefficients. In the fluid host case, it can be seen from Eqs. (15b) and (16b) that the $n=1$ scattering channel has an infinite amplitude if the conditions $\rho_{\text{eff}}^{\text{M}}=-\rho_3/2$ for solid cover and $\rho_{\text{eff}}^{\text{B}}=-\rho_3/2$ for fluid cover are satisfied. However, a similar resonance effect cannot exist in the case when the host material has nonzero shear modulus, as can be seen from Eqs. (13) and (14). Consequently, with the expressions for $\rho_{\text{eff}}^{\text{M}}$ and $\rho_{\text{eff}}^{\text{B}}$, the quasistatic resonance condition is given by

$$\frac{2\rho_2+\rho_3}{2(\rho_2-\rho_1)}=\frac{r_1^3}{r_2^3}, \tag{20}$$

for a cover material with nonzero shear modulus, and

$$\frac{(2\rho_2+\rho_3)(\rho_2+2\rho_1)}{2(\rho_2-\rho_3)(\rho_2-\rho_1)}=\frac{r_1^3}{r_2^3}, \tag{21}$$

for a cover material with zero shear modulus.

As a matter of fact, a coated sphere itself is able to exhibit a resonance if $\rho_1=-\rho_2/2$ is satisfied in the case of the fluid cover. Upon substitution of this condition into Eq. (12), it follows readily that $\rho_{\text{eff}}^{\text{B}}=\rho_1$. This implies that in the resonant state, the dynamic behavior of the inner sphere is so remarkable that the coated sphere can be fully replaced by the inner sphere. In other words, the inner sphere seems to enlarge its radius from r_1 to r_2, as if the cover material was absent. A similar phenomenon has also been found in electromagnetics and termed as the partially resonant effect.[30,31]

III. NUMERICAL RESULTS AND DISCUSSIONS

In this section, numerical results from full-wave dynamic computations are presented to discuss further the transparency phenomenon of a coated sphere embedded in a solid matrix. In a previous paper,[23] the physical mechanism and robustness of cloaking materials for acoustic transparency in a fluid system were examined. The numerical examples presented below will focus on the influence of shear modulus of each region on the transparency phenomenon, as a necessary supplement. Note that for a solid host medium, λ_3 refers to $2\pi\sqrt{\kappa_3/\rho_3}/\omega$ for better comparison of different results, where ω is the angular frequency of the incident longitudinal wave.

Consider a sphere with $\kappa_1=0.2\kappa_3$, $\mu_1=0.1\kappa_3$, and $r_1=\lambda_3/20$ covered by a material with bulk modulus $\kappa_2=-3\kappa_3$. The shear modulus of the cover will be systematically varied. The densities of the sphere, coating, and matrix are set to be equal to $\rho_1=\rho_2=\rho_3$, so that the contributions from $n=1$ scattering channel is minimized. When the coated sphere is placed in a fluid medium with $\mu_3=0$, then the contributions from $n=2$ scattering channel is eliminated. These parameter sets are employed mainly to evaluate the influence of shear modulus on the transparency occurring in the $n=0$ channel.

The effective bulk modulus κ_{eff} of the coated sphere assemblage for selected values of cloak shear modulus $\mu_2=0$, $\mu_2=0.2\kappa_3$, and $\mu_2=0.4\kappa_3$ is shown in Fig. 2(a) as a function of sphere radius ratio r_2/r_1. The effective bulk modulus is calculated with the HS bound, which naturally reduces to the Voigt bound when $\mu_2=0$. It can be seen from Fig. 2(a) that with the increase of shear modulus μ_2, the effective bulk modulus of the composite system increases. Thus, by letting κ_{eff} equal to κ_3 for the transparency, the thickness of the

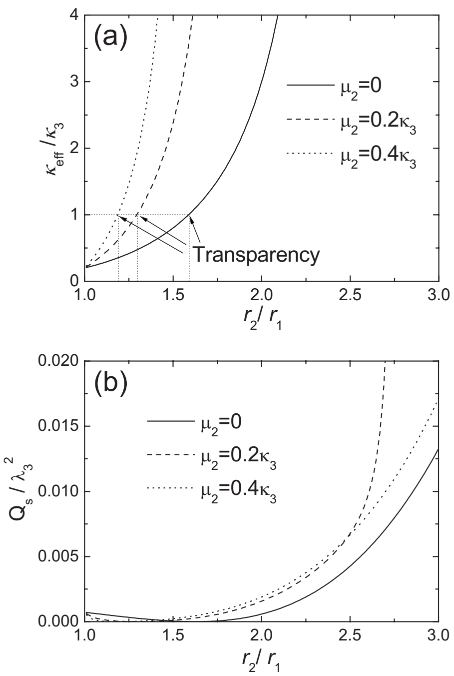

FIG. 2. (a) Effective bulk modulus κ_{eff}/κ_3 of a coated-sphere assemblage calculated with HS bound and (b) normalized total scattering cross section Q_{sca}/λ_3^2 of coated sphere as a function of ratio r_2/r_1 for selected values of cover shear modulus: $\mu_2=0$, $\mu_2=0.2\kappa_3$, and $\mu_2=0.4\kappa_3$ ($\kappa_1=0.2\kappa_3$, $\mu_1=0.1\kappa_3$, $\kappa_2=-3\kappa_3$, $\mu_3=0$, $\rho_1=\rho_2=\rho_3$, and $r_1=\lambda_3/20$).

coating should be decreased in order to reduce the volume fraction of the shell material. The corresponding total scattering cross section Q_{sca} of the coated sphere is shown in Fig. 2(b) as a function of r_2/r_1. The results of Fig. 2(b) demonstrate that low scattering occurs when the effective bulk modulus equals to the bulk modulus of the host material, verifying the transparency conditions depicted in Fig. 2(a). The low scattering "point" shifts downward when μ_2 is increased, as predicted from Fig. 2(a).

Consider next the same coated sphere as in Fig. 2(b), but with $\mu_2=0.1\kappa_3$. The effective bulk modulus κ_{eff} of the sphere embedded in a host material with varying shear mudulus ($\mu_3=0$ and $\mu_3=0.1\kappa_3$) is plotted in Fig. 3(a) as a function of r_2/r_1. Again, it is seen that the low scattering phenomena exist in both cases and can be well predicted by the given transparency conditions. It is noticed that a resonance peak takes place at about $r_2/r_1=2.1$ in the case of $\mu_3=0$, due mainly to the $n=2$ scattering channel. This is further verified in Figs. 3(b) and 3(c), which plot the first three scattering coefficients as functions of r_2/r_1 for $\mu_3=0$ and $\mu_3=0.1\kappa_3$, respectively. The resonance is induced by higher order scattering coefficients and cannot be predicted. When the host material is replaced by a solid with finite shear modulus $\mu_3=0.1\kappa_3$, the second order scattering effect is dominated by the effective shear modulus and no resonance mode is found, as can be seen from Fig. 3(c).

As another example, we investigate the scattering property of a coated sphere with the following parameter combinations: $\kappa_1=\kappa_2=\kappa_3$, $\mu_1=0.3\kappa_3$, $\mu_3=0$, $\rho_1=-0.8\rho_3$, $\rho_2=2.5\rho_3$, and $r_1=\lambda_3/20$. These parameters are selected to

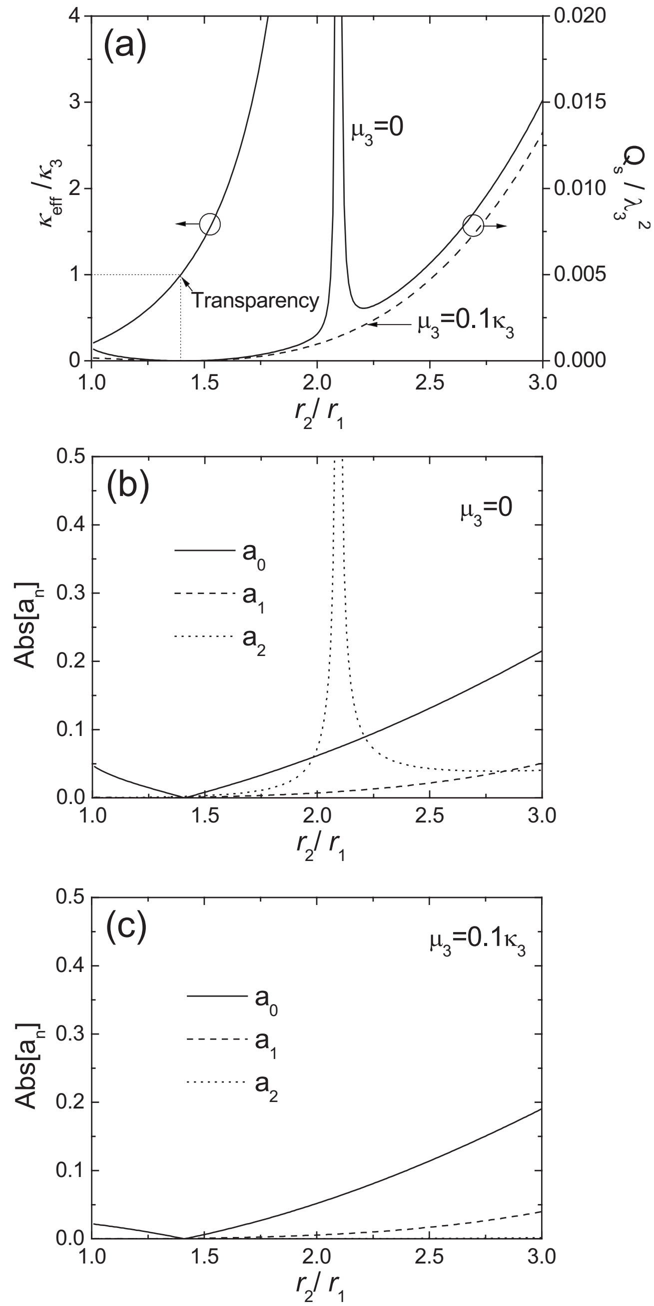

FIG. 3. (a) Effective bulk modulus κ_{eff}/κ_3 of a coated-sphere assemblage calculated with HS bound and normalized total scattering cross section Q_{sca}/λ_3^2 of coated sphere as functions of ratio r_2/r_1; contributions of first three scattering coefficients of coated sphere for (b) $\mu_3=0$ and (c) $\mu_3=0.1\kappa_3$ ($\kappa_1=0.2\kappa_3$, $\mu_1=0.1\kappa_3$, $\kappa_2=-3\kappa_3$, $\mu_2=0.1\kappa_3$, $\rho_1=\rho_2=\rho_3$, and $r_1=\lambda_3/20$).

minimize the influence of zeroth order and second order scatterings, so that the focus can be placed on the analysis of the influence of cloak shear modulus on the transparency in the $n=1$ channel. Figure 4(a) plots the effective mass density ρ_{eff} of a coated sphere assemblage computed separately with the mixing rule and Berryman's formula as a function of r_2/r_1. The results of Fig. 4(a) can be used to predict where the transparency (i.e., $\rho_{eff}=\rho_3$) and resonance (i.e., $\rho_{eff}=-\rho_3/2$) may take place. Figure 4(b) presents the total scattering cross

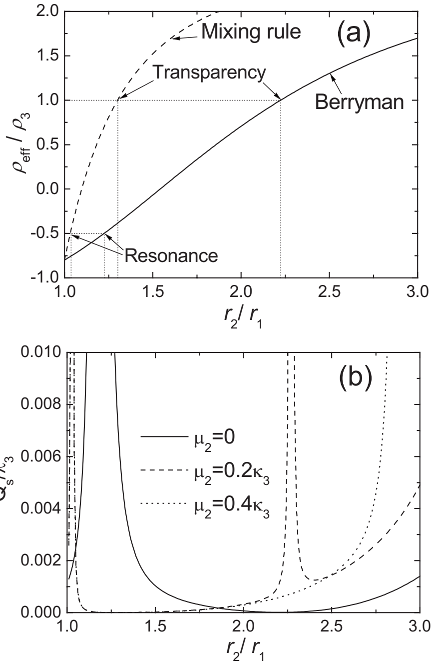

FIG. 4. (a) Effective mass density ρ_{eff}/ρ_3 of a coated-sphere assemblage calculated with mixing rule and Berryman's formula, and (b) normalized total scattering cross section $Q_{\text{sca}}/\lambda_3^2$ of coated sphere, all plotted as functions of ratio r_2/r_1 for selected values of coating shear modulus $\mu_2=0$, $\mu_2=0.2\kappa_3$, and $\mu_2=0.4\kappa_3$ ($\kappa_1=\kappa_2=\kappa_3$, $\mu_1=0.3\kappa_3$, $\mu_3=0$, $\rho_1=-0.8\rho_3$, $\rho_2=2.5\rho_3$, and $r_1=\lambda_3/20$).

section for different values of cloak shear modulus $\mu_2=0$, $\mu_2=0.2\kappa_3$, and $\mu_2=0.4\kappa_3$. Both the mixing rule and Berryman's formula give excellent predictions for the transparency and resonance phenomena. The results of Fig. 4(b) reveal that the resonance occurring at a larger cover radius for the case $\mu_2=0.2\kappa_3$ is caused by higher order scattering, which cannot be predicted with the quasistatic conditions.

For further investigations, Fig. 5(a) plots the total scattering cross section of the same coated sphere as that in Fig. 4 as a function of r_2/r_1 for the case $\mu_2=0.3\kappa_3$; the sphere is embedded in different surrounding media with $\mu_3=0$ and $\mu_3=0.3\kappa_3$. The contributions from the first three order scattering coefficients are presented in Figs. 5(b) and 5(c). It can be seen that, for both types of host medium, the transparency is achieved in channel $n=1$ and can be well predicted by the mixing rule given in Fig. 4(a). In addition, two resonant peaks that occur in the case of $\mu_3=0$ (fluid host) disappear for a solid host medium with $\mu_3=0.3\kappa_3$. The first peak with smaller cover radius vanishes because the first order natural mode does not exist for solid surroundings. The vanishing of the second peak with larger cover radius is because the second order scattering effect for a solid host is dominated by the effective shear modulus of the system, which does not support a resonance mode, as indicated by Eqs. (13d) and (13e).

Consider next a sphere with $\kappa_1=35\kappa_3$, $\mu_1=15\kappa_3$, and $\rho_1=2.7\rho_3$, representative of an aluminum sphere immersed in water.[32] When the radius of the sphere is taken to be $r_1=\lambda_3/5$, its total scattering cross section is $Q_{\text{sca}}=0.043\lambda_3^2$. A fluid cover with outer radius $r_2=1.3r_1$ is used here to cloak

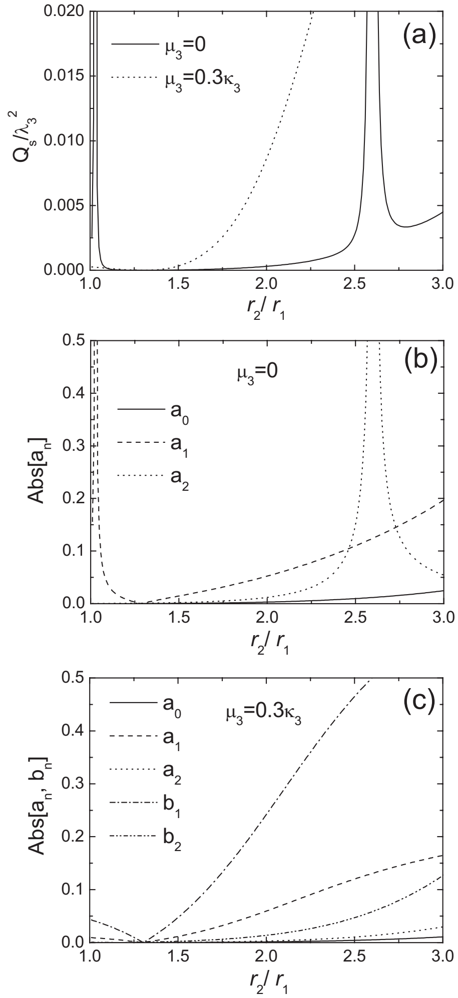

FIG. 5. (a) Normalized total scattering cross section $Q_{\text{sca}}/\lambda_3^2$ of a coated sphere plotted as a function of ratio r_2/r_1; contributions of first three scattering coefficients of the coated sphere for (b) $\mu_3=0$ and (c) $\mu_3=0.3\kappa_3$ ($\kappa_1=\kappa_2=\kappa_3$, $\mu_1=\mu_2=0.3\kappa_3$, $\rho_1=-0.8\rho_3$, $\rho_2=2.5\rho_3$, and $r_1=\lambda_3/20$).

the aluminum sphere. According to the transparency conditions (17) and (19b), the cloaking material must have a bulk modulus $\kappa_2=0.58\kappa_3$ and a mass density $\rho_2=0.55\rho_3$, which are both lower than those of water. However, the effectiveness of this cover lies within the Rayleigh limit. For the case of $r_1=\lambda_3/5$, the parameters for a minimized value of the scattering may be further tuned to the desirable values of $\kappa_2=0.47\kappa_3$ and $\rho_2=0.4\rho_3$ by varying κ_1 and ρ_1 around the target values of $\kappa_2=0.58\kappa_3$ and $\rho_2=0.55\rho_3$. As a result, the total scattering cross section of the composite sphere is reduced by 99.8% to $Q_{\text{sca}}=7.36\times10^{-4}\lambda_3^2$. Materials with these optimized parameter combinations are not readily available in nature, but can be purposely fabricated as an acoustic metamaterial. Figures 6(a) and 6(b) present the near-field contour plots of the radial component of the scattered displacement fields for a uncoated aluminum sphere and that with an optimized cloak, respectively. The scheme of the system is shown in Fig. 1 and the wave vector is along the z direction. It can be seen that a sphere without metamaterial cover leads to strong, nonuniform scattering field in the solid

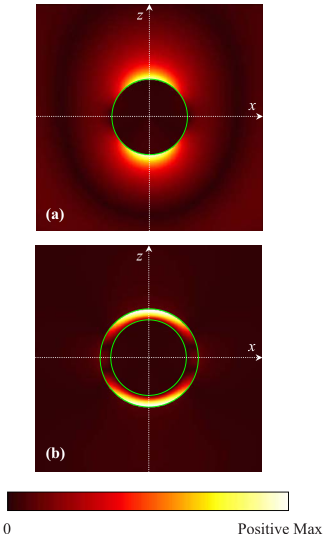

FIG. 6. (Color online) Contour plots of radial component of scattered displacement field for (a) uncoated aluminum sphere with $\rho_1=2.7\rho_3$, $\kappa_1=35\kappa_3$, $\mu_1=15\kappa_3$, and $r_1=\lambda_3/5$; (b) same sphere with fluid cover having parameters $\kappa_1=0.47\kappa_3$, $\rho_1=0.4\rho_3$, and $r_2=1.3r_1$.

matrix, especially in the region adjacent to the sphere. However, when cloaking metamaterial is employed as the cover, the scattering is dramatically reduced, while the field strength within the cloak is large. In both cases, the displacement field inside the aluminum sphere is negligibly small due to its large modulus. It is thus demonstrated that, with the cloaking metamaterial, the impenetrable sphere can indeed achieve acoustic transparency. This property of the composite system can lead to potential applications in underwater stealth technology.

Finally, for a fluid system, consider a hollow sphere with inner radius $r_1=\lambda_3/20$. The cavity of the hollow sphere is filled with the host material. The cover has the same bulk modulus as the host (i.e., $\kappa_2=\kappa_3$), but has a different mass density ρ_2. The total scattering cross section of the hollow sphere is calculated as a function of ρ_2/ρ_1 for selected values of cover radius: $r_2=1.1r_1$, $r_2=1.15r_1$, and $r_2=1.2r_1$, as plotted in Fig. 7. It can be seen from Fig. 7 that there is low scattering around $\rho_2=-2.1\rho_1$, independent of cover radius. As mentioned in Sec. II, the quasistatic partially resonate state of a coated sphere takes place at its $n=1$ scattering channel, when $\rho_2=-2\rho_1$ is satisfied. When the resonance occurs, the solid shell is absent and the coated sphere can be fully represented by the inner sphere, i.e., cavity in this case. It is evident that the transparency phenomenon shown in Fig. 7 is due to the partially resonant effect of the hollow sphere, but the low scattering point has been shifted to $\rho_2=-2.1\rho_1$ as a result of the large particle effect. When the partially resonance is excited, the hollow sphere is inherently transparent, independent of the cover thickness.

Analyses concerning the transparency condition (18) are not presented here, since matematerials with negative shear modulus have not been proposed. Normally, Eqs. (17) and

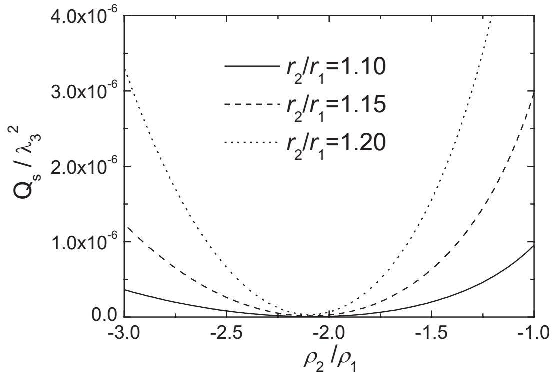

FIG. 7. Total scattering cross sections of a hollow sphere plotted as a function of ρ_2/ρ_1 for selected values of cover radius: $r_2=1.1r_1$, $r_2=1.15r_1$, and $r_2=1.2r_1$ ($\kappa_1=\kappa_2=\kappa_3$, $\mu_1=\mu_2=\mu_3=0$, $\rho_1=\rho_3$, and $r_1=\lambda_3/20$).

(18) are sufficient to predict the transparency phenomenon, since the surrounding medium of interest is often fluid material. In this case, the scattering induced by shear modulus in the Rayleigh limit always vanishes. As a result, the above analyses reveal that a coated solid sphere can be made transparent for an incident compressional wave. The cloaking materials are isotropic and can be readily manufactured with the techniques developed recently.[16–20] Hence, potential applications in the stealth technology can be anticipated.

IV. CONCLUSIONS

The elastic wave transparency phenomenon of a coated solid sphere embedded in a host solid (or fluid) medium was analytically studied for incident compressional waves. In the Rayleigh limit, the first three scattering coefficients were derived, and it is found that the coated sphere can be represented by a single effective sphere. The Hashin-Shtrikman bound, the volume averaged method, and the Berryman's formula were used to estimate the effective modulus and density parameters of the coated sphere (assemblage of the coated sphere system). Quasistatic transparency conditions were obtained by setting the first three scattering coefficients as zero, which are exactly the results predicted by the neutral inclusion concept. The resonance phenomenon and partially resonance effect of a coated sphere induced by mass densities were also studied. Numerical results with full-wave computations reveal that the given quasistatic conditions give excellent prediction of the transparency and resonance phenomena in the dynamic case.

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