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A new derivative on the shift property of effective elastic compliances for planar and three-dimensional composites

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Based on Hill's condition and a field-fluctuation approach, a new derivation is given for the shift property of effective compliances for both planar and three-dimensional composites. The derived relations are exact, and hold for any kind of microstructures and anisotropy. To provide a link to the well-known shift property of Cherkaev, Lurie & Milton in plane elasticity, special reference is given to two-dimensional composites and voided materials with isotropic constituents, but covering both overall in-plane isotropy and orthotropy. This method is substantially simpler than the stress-invariance approach commonly adopted in the literature and it provides a new means of addressing the shift characteristics of the effective compliances. By this approach, several universal relations governing the effective compliances of three-dimensional and two-dimensional composites are also found, to our knowledge, for the first time.

Keywords: composites; universal relations; shift properties; effective compliances

1. Introduction

The remarkable finding by Cherkaev, Lurie & Milton (Cherkaev *et al.* 1992) on the stress invariance and shift characteristics of effective compliances in planar elasticity has proved to have fundamental implications for the determination of effective properties of composite materials. This result, commonly referred to as the CLM theorem, has been further explored by Thorpe & Jasiuk (1992), Dundurs & Markenscoff (1993), Moran & Gosz (1994), Zheng & Hwang (1996, 1997) and Dundurs & Jasiuk (1997), suggesting that such characteristics hold for a wider range of shift conditions. This idea was also examined for planar piezoelectric and electromagnetic thermoelastic media (Chen 1995; Zheng & Chen 1999*a*, *b*), and planar Cosserat elasticity (Ostoja-Starzewski & Jasiuk 1995). The condition for stress invariance in a full three-dimensional body was recently discussed by Norris (1999). Based on the stress-invariance property, when the local compliance is properly shifted, the effective compliance tensor of a planar composite will have an identical shift, regardless of its microstructure. These universal relations can serve to validate the different micromechanical models, and can explain some of the fundamental phenomena in plane elasticity, such as Michell's (1899) theorem, Dundurs's (1967) constants, and why the in-plane Young's modulus of a voided plate is independent of the Poisson's ratio of the matrix.

The foregoing derivations on the shift characteristics are generally based on consideration of the stress-invariance condition that in turn leads to consideration of the compatibility condition of the displacement field. As such, a set of partial differential equations has to be examined in the light of plane elasticity. In this paper, we provide an alternative means of studying the shift characteristics of the composite without appealing to the stress-invariance condition. The method is based directly on Hill's condition and a field-fluctuation approach. Some new results will also be given.

2. Hill's condition and field fluctuation

In order to establish the effective property of a composite material, a representative volume element (RVE) is introduced as an intermediate passage from the microscopic to the macroscopic scale. More detailed discussions concerning this point can be found in Nemat-Nasser & Hori (1993). The boundary of the RVE can be either a uniform stress $\bar{\boldsymbol{\sigma}}$ or a uniform strain $\bar{\boldsymbol{\varepsilon}}$ (boldface letters denote tensors). For any statistically admissible stress field $\boldsymbol{\sigma}$ and kinematically admissible strain field $\boldsymbol{\varepsilon}$, it has $\langle \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \rangle = \langle \boldsymbol{\sigma} \rangle : \langle \boldsymbol{\varepsilon} \rangle$, where $\langle \cdot \rangle$ denotes the volume average of the said quantity over the entire RVE. This property is usually called Hill's condition (Hill 1963). Hill's condition ensures the unique definition of the effective elastic property of the composite, regardless of the chosen boundary condition.

Without loss of generality we shall consider the uniform-stress boundary condition so that the results derived can be readily connected to the existing ones. It follows that

$$\langle \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \rangle = \bar{\boldsymbol{\sigma}} : \bar{\boldsymbol{\varepsilon}}, \tag{2.1}$$

where $\bar{\boldsymbol{\varepsilon}} = \langle \boldsymbol{\varepsilon} \rangle$. With the help of the effective compliance tensor $\boldsymbol{M}$ and the local compliance tensor $\boldsymbol{m}$, equation (2.1) can be rearranged as

$$\langle \boldsymbol{\sigma} : \boldsymbol{m} : \boldsymbol{\sigma} \rangle = \bar{\boldsymbol{\sigma}} : \boldsymbol{M} : \bar{\boldsymbol{\sigma}}. \tag{2.2}$$

Now, following the field-fluctuation method advanced by Bobeth & Diener (1987) and Kreher & Pompe (1989) in the RVE (see also Hu 1996), let the local compliance tensor have a variation $\delta\boldsymbol{m}$, while keeping the applied stress $\bar{\boldsymbol{\sigma}}$ fixed. Equation (2.2) then leads to

$$2\langle \boldsymbol{\sigma} : \boldsymbol{m} : \delta\boldsymbol{\sigma} \rangle + \langle \boldsymbol{\sigma} : \delta\boldsymbol{m} : \boldsymbol{\sigma} \rangle = \bar{\boldsymbol{\sigma}} : \delta\boldsymbol{M} : \bar{\boldsymbol{\sigma}}. \tag{2.3}$$

Since $\boldsymbol{\sigma} : \boldsymbol{m}$ is a kinematically admissible strain field and $\delta\boldsymbol{\sigma}$ is a statistically admissible stress field with $\langle \delta\boldsymbol{\sigma} \rangle = 0$, it can be shown using Hill's condition again that $\langle \boldsymbol{\sigma} : \boldsymbol{m} : \delta\boldsymbol{\sigma} \rangle = 0$. So for any variation of the local compliance tensor $\delta\boldsymbol{m}$, the following relation holds

$$\langle \boldsymbol{\sigma} : \delta\boldsymbol{m} : \boldsymbol{\sigma} \rangle = \bar{\boldsymbol{\sigma}} : \delta\boldsymbol{M} : \bar{\boldsymbol{\sigma}}. \tag{2.4}$$

This relation is universal, and it will serve as the basis for the new derivation of shift characteristics of the effective compliance tensor.

3. Shift property of the effective compliances

(*a*) *A universal relation for a three-dimensional composite*

Now let the local compliance tensor have the following variation $\mathrm{d}\boldsymbol{m} = \mathrm{d}k\boldsymbol{m}_0$, where $\boldsymbol{m}_0$ is the local compliance tensor before the variation. Equation (2.4) becomes

$$\langle \boldsymbol{\sigma} : \mathrm{d}k\boldsymbol{m}_0 : \boldsymbol{\sigma} \rangle = \mathrm{d}k\langle \boldsymbol{\sigma} : \boldsymbol{m}_0 : \boldsymbol{\sigma} \rangle = \bar{\boldsymbol{\sigma}} : \mathrm{d}\boldsymbol{M} : \bar{\boldsymbol{\sigma}}. \tag{3.1}$$

Thus,

$$\langle \boldsymbol{\sigma} : \boldsymbol{m}_0 : \boldsymbol{\sigma} \rangle = \bar{\boldsymbol{\sigma}} : \boldsymbol{M}_0 : \bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}} : \frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}k} : \bar{\boldsymbol{\sigma}}, \tag{3.2}$$

for any applied $\bar{\boldsymbol{\sigma}}$. It follows that

$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}k} = \boldsymbol{M}_0. \tag{3.3}$$

In a more direct form, equation (3.3) also means that if the effective compliance tensor is written as $\boldsymbol{M} = \boldsymbol{\Psi}(\boldsymbol{m}, \boldsymbol{\Xi})$, then $k\boldsymbol{M} = \boldsymbol{\Psi}(k\boldsymbol{m}, \boldsymbol{\Xi})$ for any $k > 0$, where $\boldsymbol{\Xi}$ represents the microstructural features. This relation is valid for any anisotropy and microstructure of the composite, and it also implies that $\boldsymbol{M}$ is a homogeneous function of $\boldsymbol{m}$. Such a homogeneity connection has also been found by Milgrom (1990) for general linear phenomena from a different perspective.

(*b*) *Planar composites*

Consider a planar composite under either the plane-stress or the plane-strain condition whose in-plane properties (1–2 plane) are of interest. Its local and effective compliance tensors are still denoted by $\boldsymbol{m}$ and $\boldsymbol{M}$, respectively. Now let the variation of the local compliance tensor carries the form $\mathrm{d}\boldsymbol{m} = \mathrm{d}\lambda\boldsymbol{J}$, where

$$J_{ijkl} = \delta_{ij}\delta_{kl} - \tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

and i, j, k and l range from 1 to 2. Equation (2.4) leads to

$$\langle \boldsymbol{\sigma} : \boldsymbol{J} : \boldsymbol{\sigma} \rangle = \bar{\boldsymbol{\sigma}} : \frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}\lambda} : \bar{\boldsymbol{\sigma}}. \tag{3.4}$$

Furthermore, we note that, in planar elasticity, the following identities exist

$$\begin{aligned}\langle \boldsymbol{\sigma} : \boldsymbol{J} : \boldsymbol{\sigma} \rangle &= 2\langle \sigma_{11}\sigma_{22} - \sigma_{12}\sigma_{12} \rangle = 2\langle \sigma_{11}\varphi,_{11} + \sigma_{12}\varphi,_{12} \rangle \\ &= 2(\bar{\sigma}_{11}\bar{\sigma}_{22} - \bar{\sigma}_{12}\bar{\sigma}_{12}) = \bar{\boldsymbol{\sigma}} : \boldsymbol{J} : \bar{\boldsymbol{\sigma}},\end{aligned} \tag{3.5}$$

where the equilibrium condition, Gauss's divergence theorem, and Airy's stress function φ have been used. Immediately one finds

$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}\lambda} = \boldsymbol{J}. \tag{3.6}$$

This relation holds for any anisotropy and any microstructure of the planar composite. It also implies that

$$\boldsymbol{M} - \lambda\boldsymbol{J} = \boldsymbol{\Psi}(\boldsymbol{m} - \lambda\boldsymbol{J}, \boldsymbol{\Xi}) \tag{3.7}$$

for any allowed λ (keep $\boldsymbol{m} - \lambda\boldsymbol{J}$ positive definite). This general anisotropic shift characteristic is quite similar to the one derived by Zheng & Hwang (1996). In its isotropic form, this relation is exactly the one derived by Cherkaev *et al.* (1992) when the area bulk modulus and shear modulus of the field possess a constant but opposite shift. That is, if at the field point

$$\frac{1}{k} \to \frac{1}{k} - 2\lambda \quad \text{and} \quad \frac{1}{\mu} \to \frac{1}{\mu} + 2\lambda, \tag{3.8}$$

then, for the planar composite,

$$\frac{1}{k_\mathrm{c}} \to \frac{1}{k_\mathrm{c}} - 2\lambda \quad \text{and} \quad \frac{1}{\mu_\mathrm{c}} \to \frac{1}{\mu_\mathrm{c}} + 2\lambda, \tag{3.9}$$

where k and μ are the local area bulk and shear moduli, respectively, and k_c and μ_c are those of the overall composite.

(*c*) *Voided or cracked planar materials*

Here we specialize on the voided or cracked solids in an isotropic matrix with Young's modulus E_m and Poisson's ratio ν_m. The geometry of voids or cracks is left as arbitrary, but the crack surfaces are taken to have no closure effect. Under plane stress the in-plane elastic compliance tensor of the matrix can be written as $\boldsymbol{m} = (1/E_\mathrm{m})(\boldsymbol{I} - \nu_\mathrm{m}\boldsymbol{J})$, where $I_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ and i, j, k and l range from 1 to 2. Under plane strain, E_m and ν_m should be replaced by E'_m and ν'_m, where

$$E'_\mathrm{m} = \frac{E_\mathrm{m}}{1-\nu_\mathrm{m}^2}, \qquad \nu'_\mathrm{m} = \frac{\nu_\mathrm{m}}{1-\nu_\mathrm{m}}.$$

Now, for plane stress, equation (2.4) translates into

$$(1-c)\langle \mathrm{d}(1/E_\mathrm{m})\boldsymbol{\sigma} : \boldsymbol{\sigma} - \mathrm{d}(\nu_\mathrm{m}/E_\mathrm{m})\boldsymbol{\sigma} : \boldsymbol{J} : \boldsymbol{\sigma}\rangle_\mathrm{m} = \bar{\boldsymbol{\sigma}} : \mathrm{d}\boldsymbol{M} : \bar{\boldsymbol{\sigma}}, \tag{3.10}$$

where c is the volume concentration of the voids. Now let E_m be kept fixed and ν_m have a variation $\mathrm{d}\nu_\mathrm{m}$. With the help of the identity (3.5), or

$$(1-c)\langle \boldsymbol{\sigma} : \boldsymbol{J} : \boldsymbol{\sigma}\rangle_\mathrm{m} = \langle \boldsymbol{\sigma} : \boldsymbol{J} : \boldsymbol{\sigma}\rangle = \bar{\boldsymbol{\sigma}} : \boldsymbol{J} : \bar{\boldsymbol{\sigma}},$$

equation (3.10) leads to

$$\bar{\boldsymbol{\sigma}} : \left(\frac{\partial \boldsymbol{M}}{\partial \nu_\mathrm{m}} + \frac{1}{E_\mathrm{m}}\boldsymbol{J}\right) : \bar{\boldsymbol{\sigma}} = 0, \tag{3.11}$$

and thus

$$\frac{\partial \boldsymbol{M}}{\partial \nu_\mathrm{m}} + \frac{1}{E_\mathrm{m}}\boldsymbol{J} = 0, \tag{3.12}$$

for any planar voided body.

When the effective in-plane property of the voided medium is isotropic, with $\boldsymbol{M} = (1/E_\mathrm{c})(\boldsymbol{I} - \nu_\mathrm{c}\boldsymbol{J})$, equation (3.12) can be decomposed into

$$\frac{\partial(1/E_\mathrm{c})}{\partial \nu_\mathrm{m}} = 0, \qquad \frac{\partial(\nu_\mathrm{c}/E_\mathrm{c})}{\partial \nu_\mathrm{m}} = \frac{1}{E_\mathrm{m}}. \tag{3.13}$$

The first equation implies that the effective Young's modulus of the voided material is independent of the Poisson ratio of the matrix, as demonstrated numerically by Day *et al.* (1992) and already proved by the CLM theorem. With this property, the second equation further suggests that the effective Poisson's ratio of the voided material ν_{c} is a linear function of the Poisson ratio of the matrix ν_{m}, with the coefficient $E_{\mathrm{c}}/E_{\mathrm{m}}$. To our knowledge, this relation is also new.

If the distribution of voids is such that the voided material is rendered orthotropic, with the in-plane Young's moduli E_1 and E_2, Poisson's ratio ν_{12}, and shear modulus G_{12}, equation (3.12) leads to

$$\frac{\partial(1/E_1)}{\partial\nu_{\mathrm{m}}}=0,\qquad \frac{\partial(1/E_2)}{\partial\nu_{\mathrm{m}}}=0,\qquad \frac{\partial(\nu_{12}/E_1)}{\partial\nu_{\mathrm{m}}}=\frac{1}{E_{\mathrm{m}}},\qquad \frac{\partial(1/2G_{12})}{\partial\nu_{\mathrm{m}}}=\frac{1}{E_{\mathrm{m}}}. \tag{3.14}$$

Thus both Young's moduli are independent of the matrix Poisson ratio, but not the shear modulus G_{12}, whose compliance $1/G_{12}$, like ν_{12}, has a linear dependence on ν_{m}. These orthotropic shift characteristics appear not to have been reported previously.

The foregoing conclusions still hold for plane strain, but the Young's modulus and Poisson's ratio of the matrix, as well as the effective Young's modulus, E_{c}, and Poisson's ratio, ν_{c}, of the composite, should be interpreted in the plane-strain context. The connections between the in-plane E_{c}, ν_{c}, k_{c} and μ_{c} and the elastic constants of the full three-dimensional Hooke's law of a transversely isotropic solid depend on the plane-stress or plane-strain condition. For instance if the 1–2 plane is isotropic with $E_1=E_2=E_{\mathrm{T}}$ and $\nu_{12}=\nu_{21}=\nu_{\mathrm{T}}$ (subscript 'T' denotes the transverse direction), the relations for plane stress are simply

$$E_{\mathrm{c}}=E_{\mathrm{T}},\qquad k_{\mathrm{c}}=\frac{E_{\mathrm{T}}}{2(1-\nu_{\mathrm{T}})},\qquad \nu_{\mathrm{c}}=\nu_{\mathrm{T}},\qquad \mu_{\mathrm{c}}=\frac{E_{\mathrm{T}}}{2(1+\nu_{\mathrm{T}})}, \tag{3.15}$$

but, for plane strain,

$$\left.\begin{aligned} E_{\mathrm{c}}&=\frac{E_{\mathrm{T}}}{1-\nu_{13}\nu_{31}}, & k_{\mathrm{c}}&=\frac{E_{\mathrm{T}}}{2(1-\nu_{\mathrm{T}}-2\nu_{13}\nu_{31})},\\ \nu_{\mathrm{c}}&=\frac{\nu_{\mathrm{T}}+\nu_{13}\nu_{31}}{1-\nu_{13}\nu_{31}}, & \mu_{\mathrm{c}}&=\frac{E_{\mathrm{T}}}{2(1+\nu_{\mathrm{T}})},\end{aligned}\right\} \tag{3.16}$$

where the Poisson's ratio ν_{13} refers to strain induced in the 3-direction due to a tensile stress in the 1-direction, and vice versa for ν_{31}. Then, for plane strain, E_{c} and ν_{c} are those in (3.16) and E_{m} and ν_{m} are replaced by E'_{m} and ν'_{m}, respectively. Furthermore, these relations apply to the transversely isotropic matrix (in the 1–2 plane) when its E_{m} and ν_{m} or E'_{m} and ν'_{m} are interpreted in the plane-stress or plane-strain context as in (3.15) and (3.16).

We conclude this section by looking as an example at the Hashin–Shtrikman (H–S) (Hashin & Shtrikman 1963) upper and lower bounds for the plane-strain bulk and shear moduli of a transversely isotropic two-dimensional composite (Hashin 1965). First, for a porous material its upper bounds are

$$\frac{1}{k_{\mathrm{c}}}=\frac{1}{1-c}\frac{1}{k_{\mathrm{m}}}+\frac{c}{1-c}\frac{1}{\mu_{\mathrm{m}}},\qquad \frac{1}{\mu_{\mathrm{c}}}=\frac{2c}{1-c}\frac{1}{k_{\mathrm{m}}}+\frac{1+c}{1-c}\frac{1}{\mu_{\mathrm{m}}}. \tag{3.17}$$

It is then easy to see that this pair of moduli satisfies the shift property of (3.8) and (3.9). Furthermore, since $E_{\mathrm{c}}=2(1-\nu_{\mathrm{c}})k_{\mathrm{c}}=2(1+\nu_{\mathrm{c}})\mu_{\mathrm{c}}$, or $\nu_{\mathrm{c}}=(k_{\mathrm{c}}-\mu_{\mathrm{c}})/(k_{\mathrm{c}}+\mu_{\mathrm{c}})$ and $1/E_{\mathrm{c}}=\frac{1}{4}(1/k_{\mathrm{c}}+1/\mu_{\mathrm{c}})$ regardless of the plane stress or plane strain, one finds that the plane-strain Poisson's ratio derived from (3.17),

$$\nu_{\mathrm{c}} = \frac{c + (1-c)\nu_{\mathrm{m}}}{1+2c}, \tag{3.18}$$

is indeed a linear function of ν_{m}, whose coefficient $(1-c)/(1+2c)$ is exactly the ratio $E_{\mathrm{c}}/E_{\mathrm{m}}$, as envisioned from $(3.13)_2$. Though more elaborate to demonstrate, we have also checked that both the Hashin–Shtrikman upper and lower bounds of a general two-dimensional transversely isotropic composite also satisfy the CLM shift property.

4. Differential equations for the effective compliance tensor with isotropic constituents

In this section we derive a new set of differential equations that must be satisfied by the effective compliance tensor $\boldsymbol{M}$ of a composite or voided body with isotropic constituents. The composite as a whole can have any microstructure and anisotropy. Both three-dimensional and two-dimensional composites will be considered.

(a) *Three-dimensional solids*

First, we note that the energy density of the isotropic constituent i, denoted by w_i, can be written in the following form

$$w_i = a_i\sigma^2 + b_i\boldsymbol{s} : \boldsymbol{s}, \tag{4.1}$$

where σ is the isotropic part of the stress tensor $\boldsymbol{\sigma}$, defined as $\sigma = \frac{1}{3}\sigma_{kk}$ $(k = 1, 2, 3)$, and $\boldsymbol{s}$ is the deviatoric part. Constants a_i and b_i are related to the bulk and shear moduli κ_i and μ_i through $a_i = 1/(2\kappa_i)$ and $b_i = 1/(4\mu_i)$ (note that κ is used for the three-dimensional bulk modulus and k is for the two-dimensional area bulk modulus).

Now let a_i and b_i have independent variations δa_i and δb_i. Noting that (2.2) is twice the elastic energy, we have from the universal relation (2.4)

$$2c_i\delta a_i\langle\sigma^2\rangle_i = \bar{\boldsymbol{\sigma}} : \delta\boldsymbol{M} : \bar{\boldsymbol{\sigma}}, \qquad 2c_i\delta b_i\langle\boldsymbol{s} : \boldsymbol{s}\rangle_i = \bar{\boldsymbol{\sigma}} : \delta\boldsymbol{M} : \bar{\boldsymbol{\sigma}}, \tag{4.2}$$

where c_i is the volume concentration of phase i and $\langle\cdot\rangle_i$ denotes the volume average of the said quantity over the ith phase. It follows that

$$2c_i\langle\sigma^2\rangle_i = \bar{\boldsymbol{\sigma}} : \frac{\partial\boldsymbol{M}}{\partial a_i} : \bar{\boldsymbol{\sigma}}, \qquad 2c_i\langle\boldsymbol{s} : \boldsymbol{s}\rangle_i = \bar{\boldsymbol{\sigma}} : \frac{\partial\boldsymbol{M}}{\partial b_i} : \bar{\boldsymbol{\sigma}}. \tag{4.3}$$

Since, for a composite with isotropic constituents,

$$\begin{aligned}\langle\boldsymbol{\sigma} : \boldsymbol{\varepsilon}\rangle &= 2\sum_i[c_ia_i\langle\sigma^2\rangle_i + c_ib_i\langle\boldsymbol{s} : \boldsymbol{s}\rangle_i] \\ &= \bar{\boldsymbol{\sigma}} : \left[\sum_i a_i\frac{\partial\boldsymbol{M}}{\partial a_i} + b_i\frac{\partial\boldsymbol{M}}{\partial b_i}\right] : \bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}} : M : \bar{\boldsymbol{\sigma}},\end{aligned} \tag{4.4}$$

we arrive at the differential equation for the effective compliance tensor

$$\sum_i\left[a_i\frac{\partial\boldsymbol{M}}{\partial a_i} + b_i\frac{\partial\boldsymbol{M}}{\partial b_i}\right] = \boldsymbol{M}, \tag{4.5}$$

of a general three-dimensional composite. The property of this differential equation indicates that $\boldsymbol{M}$ must be a homogeneous function of the variables a_i and b_i. We can therefore conclude that (4.5) is equivalent to the shift property established in (3.3), but this time the bulk and shear compliances of the constituents are allowed to shift independently.

In particular, when the overall composite is isotropic, with the bulk and shear moduli κ_c and μ_c, equation (4.5) can be decomposed into

$$\sum_i \left[\frac{1}{\kappa_i}\frac{\partial(1/\kappa_c)}{\partial(1/\kappa_i)} + \frac{1}{\mu_i}\frac{\partial(1/\kappa_c)}{\partial(1/\mu_i)}\right] = \frac{1}{\kappa_c}, \qquad \sum_i \left[\frac{1}{\mu_i}\frac{\partial(1/\mu_c)}{\partial(1/\mu_i)} + \frac{1}{\kappa_i}\frac{\partial(1/\mu_c)}{\partial(1/\kappa_i)}\right] = \frac{1}{\mu_c}. \tag{4.6}$$

Alternatively, it can also be cast in terms of moduli

$$\kappa_c = \sum_i \left[\kappa_i \frac{\partial \kappa_c}{\partial \kappa_i} + \mu_i \frac{\partial \kappa_c}{\partial \mu_i}\right], \qquad \mu_c = \sum_i \left[\mu_i \frac{\partial \mu_c}{\partial \mu_i} + \kappa_i \frac{\partial \mu_c}{\partial \kappa_i}\right]. \tag{4.7}$$

For an isotropic porous material regardless of the void shape, be it ellipsoidal, triangular, or any combination of odd shapes, the bulk and shear moduli will satisfy

$$\frac{1}{\kappa_m}\frac{\partial(1/\kappa_c)}{\partial(1/\kappa_m)} + \frac{1}{\mu_m}\frac{\partial(1/\kappa_c)}{\partial(1/\mu_m)} = \frac{1}{\kappa_c}, \qquad \frac{1}{\mu_m}\frac{\partial(1/\mu_c)}{\partial(1/\mu_m)} + \frac{1}{\kappa_m}\frac{\partial(1/\mu_c)}{\partial(1/\kappa_m)} = \frac{1}{\mu_c}. \tag{4.8}$$

This requirement is seen to be met by, for instance, the Hashin–Shtrikman upper bounds of a three-dimensional porous material

$$\frac{1}{\kappa_c} = \frac{1}{1-c}\frac{1}{\kappa_m} + \frac{3c}{4(1-c)}\frac{1}{\mu_m}, \qquad \frac{1}{\mu_c} = \frac{3+2c}{3(1-c)}\frac{1}{\mu_m} + \frac{20c}{3(1-c)}\frac{1}{9\kappa_m + 8\mu_m}. \tag{4.9}$$

We have also checked that both the H–S upper and lower bounds of a three-dimensional isotropic composite do satisfy the requirements of (4.6). To our knowledge, this pair of relations for a general three-dimensional composite is also new.

(*b*) *Planar composites*

For a planar composite with isotropic constituents, we write $\boldsymbol{\sigma} = \frac{1}{2}\sigma_{kk}$ $(k = 1, 2)$ and $a_i = 1/(2k_i)$, with the area bulk modulus $k_i = E_i/[2(1-\nu_i)]$ for plane stress and $E_i/[2(1+\nu_i)(1-2\nu_i)]$ for plane strain. Since

$$\bar{\boldsymbol{\sigma}} : \boldsymbol{J} : \bar{\boldsymbol{\sigma}} = \langle \boldsymbol{\sigma} : \boldsymbol{J} : \boldsymbol{\sigma} \rangle = \langle 2\sigma^2 - \boldsymbol{s} : \boldsymbol{s} \rangle,$$

one finds, with (4.3), that

$$\bar{\boldsymbol{\sigma}} : \left[\sum_i 2\frac{\partial \boldsymbol{M}}{\partial a_i} - \frac{\partial \boldsymbol{M}}{\partial b_i}\right] : \bar{\boldsymbol{\sigma}} = 2\bar{\boldsymbol{\sigma}} : \boldsymbol{J} : \bar{\boldsymbol{\sigma}}. \tag{4.10}$$

Thus, the effective compliance tensor of a planar composite satisfies the simple form

$$\sum_i \left[2\frac{\partial \boldsymbol{M}}{\partial a_i} - \frac{\partial \boldsymbol{M}}{\partial b_i}\right] = 2\boldsymbol{J}. \tag{4.11}$$

When the composite is transversely isotropic, it can be decomposed into

$$\sum_i \left[\frac{\partial(1/k_c)}{\partial(1/k_i)} - \frac{\partial(1/k_c)}{\partial(1/\mu_i)}\right] = 1, \qquad \sum_i \left[\frac{\partial(1/\mu_c)}{\partial(1/\mu_i)} - \frac{\partial(1/\mu_c)}{\partial(1/k_i)}\right] = 1, \tag{4.12}$$

or, alternatively,

$$k_{\mathrm{c}}^{2}=\sum_{i}\left[k_{i}^{2}\frac{\partial k_{\mathrm{c}}}{\partial k_{i}}-\mu_{i}^{2}\frac{\partial k_{\mathrm{c}}}{\partial \mu_{i}}\right],\qquad \mu_{\mathrm{c}}^{2}=\sum_{i}\left[\mu_{i}^{2}\frac{\partial \mu_{\mathrm{c}}}{\partial \mu_{i}}-\kappa_{i}^{2}\frac{\partial \mu_{\mathrm{c}}}{\partial k_{i}}\right]. \tag{4.13}$$

These simple relations provide the constraints that the area bulk and shear moduli of a multiphase planar composite must satisfy. These relations are also new, and are satisfied by the two-dimensional Hashin–Shtrikman bounds. It is also plain to see from (4.12) that, if $\mathrm{d}(1/k_i)=-\mathrm{d}(1/\mu_i)=\mathrm{d}\lambda$ for all i, then $\mathrm{d}(1/k_{\mathrm{c}})=-\mathrm{d}(1/\mu_{\mathrm{c}})=\mathrm{d}\lambda$ for the composite. This is yet another way of arriving at the CLM shift property as stated in (3.8) and (3.9).

When applied to a two-dimensional voided medium, this pair reduces to

$$\frac{\partial(1/k_{\mathrm{c}})}{\partial(1/k_{\mathrm{m}})}-\frac{\partial(1/k_{\mathrm{c}})}{\partial(1/\mu_{\mathrm{m}})}=1,\qquad \frac{\partial(1/\mu_{\mathrm{c}})}{\partial(1/\mu_{\mathrm{m}})}-\frac{\partial(1/\mu_{\mathrm{c}})}{\partial(1/k_{\mathrm{m}})}=1. \tag{4.14}$$

In addition we may multiply (4.11) by $-(\frac{1}{2}a_{\mathrm{m}}-b_{\mathrm{m}})$ and combine it with (4.5) to get

$$\tfrac{1}{2}(\tfrac{1}{2}a_{\mathrm{m}}+b_{\mathrm{m}})\left(2\frac{\partial \boldsymbol{M}}{\partial a_{\mathrm{m}}}+\frac{\partial \boldsymbol{M}}{\partial b_{\mathrm{m}}}\right)+(\tfrac{1}{2}a_{\mathrm{m}}-b_{\mathrm{m}})\boldsymbol{J}=\boldsymbol{M}, \tag{4.15}$$

as another constraint for $\boldsymbol{M}$. When cast for a transversely isotropic porous medium, this relation translates into the pair

$$\left.\begin{aligned}\frac{1}{2}\left(\frac{1}{k_{\mathrm{m}}}+\frac{1}{\mu_{\mathrm{m}}}\right)\left[\frac{\partial(1/k_{\mathrm{c}})}{\partial(1/k_{\mathrm{m}})}+\frac{\partial(1/k_{\mathrm{c}})}{\partial(1/\mu_{\mathrm{m}})}\right]+\frac{1}{2}\left(\frac{1}{k_{\mathrm{m}}}-\frac{1}{\mu_{\mathrm{m}}}\right)=\frac{1}{k_{\mathrm{c}}},\\ \frac{1}{2}\left(\frac{1}{k_{\mathrm{m}}}+\frac{1}{\mu_{\mathrm{m}}}\right)\left[\frac{\partial(1/\mu_{\mathrm{c}})}{\partial(1/k_{\mathrm{m}})}+\frac{\partial(1/\mu_{\mathrm{c}})}{\partial(1/\mu_{\mathrm{m}})}\right]+\frac{1}{2}\left(\frac{1}{\mu_{\mathrm{m}}}-\frac{1}{k_{\mathrm{m}}}\right)=\frac{1}{\mu_{\mathrm{c}}}.\end{aligned}\right\} \tag{4.16}$$

When k_{c} and μ_{c} are given by the upper bounds (3.17), one can immediately verify that (4.16) is satisfied. While for a three-dimensional porous medium only one pair of relations (4.8) is found, for the two-dimensional medium two pairs, (4.14) and (4.15), are available, the first one being connected to the CLM translation that is not available to a three-dimensional composite.

(c) *Final remarks*

Based on Hill's condition and a field-fluctuation method, we have developed a new and simple approach to derive the shift characteristics of effective compliances of both two-dimensional and three-dimensional composites. The results are exact, and independent of microgeometries and anisotropy. Along the way, the classical CLM theorem has been recovered and many new shift relations have been found. It has also been demonstrated that the Hashin–Shtrikman bounds satisfy all these relations. This is expected, since, for a well-ordered composite $((\kappa_2-\kappa_1)(\mu_2-\mu_1)\geqslant 0)$, the H–S bounds are realizable (Norris 1985; Milton 1986; Francfort & Murat 1986). But other micromechanical models whose microstructures are not known must be tested against the derived characteristics in order to prove their value.

It is worthwhile to note that dual relations for the effective stiffness can also be derived from the uniform strain boundary condition. Due to the unique nature of the effective property, however, the modulus shift characteristics can also be found from the compliance shift characteristics, such as (4.7) versus (4.6).

Finally, it should be recognized that the CLM theorem, as well as others mentioned in § 1, also addressed the issue of local stress invariance. As such, they did address a wider issue of elasticity. This is a subject that has not been considered here, and, at this moment, it is not clear whether such an issue can be addressed by the proposed approach.

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