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A numerical method for designing acoustic cloak with arbitrary shapes

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In this paper, we propose a numerical method to evaluate the material parameters of acoustic cloaks with arbitrary shapes. The transformation of the cloak layer assumed to be harmonic, so it can be uniquely determined by Laplace's equation together with proper boundary conditions. The material parameters in the transformed space are then calculated based on the relation between the pure stretches in deformation and mass density as well as bulk modulus. The calculation of the transformed material parameters and the numerical validation for an arbitrary acoustic cloak can be well integrated in a two-step model with help of the finite-element software COMSOL Multiphysics. Numerical examples are provided to illustrate these ideas.

1. Introduction

Electromagnetic metamaterials [1,2] usually refer to man-made materials which have effective electromagnetic property difficult found in nature materials. They provide an unprecedented possibility to control wave propagation, leading to very interesting devices such as cloak [3], rotators [4], concentrators [5], beam shifter and splitter [6]. As an acoustic analogue of the electromagnetic metamaterial, acoustic metamaterials have been found to possess negative effective mass density and/or negative effective bulk modulus [7]. Similarly, some interesting acoustic devices can be designed based on the form invariance of Helmholtz equation and the coordinate transformation method [3,8–11]. The acoustic cloak, that can make itself and the cloaked region transparent to acoustic wave, is also an active research field. Chen and Chan [12] firstly obtain the general transformation expression of acoustic material parameters by using the invariant property of the acoustic equation. Their important work makes the idea of transformation acoustics (TA) possible. The derived material parameters for a 3D spherical cloak in [12] are further confirmed by Cummer et al. [13] from the scattering theory. The material parameters for a 2D cylindrical acoustic cloak have been derived by Cummer and Schurig [14], which is the special case of the results given by Chen and Chan [12]. Greenleaf et al. [15] point out that the results in [12,13] follow directly from a full wave analysis of cloaking for Helmholtz equation with respect to Riemannian metrics [[16], Sec. 3]. Recently, Norris [17] shows that for a given transformation mapping, the material composition of an acoustic cloak is not uniquely defined, thus opening up a vast range of potential materials for realizing acoustic cloaks. For elastic wave, Milton et al. [18] show that Willis equations are form invariance under a coordinate transformation, which suggest a route for realizing elastic wave cloaks.

Although several methods have been proposed for designing arbitrary electromagnetic cloaks [19], due to the different governing equations and the different form of the transformed material parameters between acoustic cloak and electromagnetic cloak, to date, the design method for an arbitrary cloak based on Helmholtz equation is not available. The objective of this paper is to propose a numerical method for evaluating the transformed material parameters of an acoustic cloak with arbitrary shape. The paper will be arranged as follows: the design method will be explained in Section 2, the application of the proposed method to design an arbitrary acoustic cloak will be presented in Section 3, followed by discussion and conclusion.

2. Transformation method and related deformation

The coordinate transformation method for designing acoustic cloaks is based on the form invariance of acoustic equation (Helmholtz equation), which is written as

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$$\nabla \cdot (\boldsymbol{\rho}^{-1}\nabla p) + \omega^2\kappa^{-1}p = 0. \tag{1}$$

Under a mapping transformation

$$\mathbf{x}' = \mathbf{x}'(\mathbf{x}), \quad p'(\mathbf{x}') = p(\mathbf{x}), \tag{2}$$

Eq. (1) will retain its form, but be rewritten as

$$\nabla' \cdot (\boldsymbol{\rho}'^{-1}\nabla' p') + \omega^2\kappa'^{-1}p' = 0, \tag{3}$$

where the material parameters are given by Ref. [12]

$$\boldsymbol{\rho}'^{-1} = \mathbf{A}\boldsymbol{\rho}^{-1}\mathbf{A}^{\mathrm{T}}/\det\mathbf{A}, \quad \kappa' = \kappa \det\mathbf{A}. \tag{4}$$

where $\mathbf{A}$ is the Jacobian transformation tensor with components $A_{ij} = \partial x'_i/\partial x_j$, it characterizes the mapping from the original space Ω to the transformed space Ω' .

It can be found from Eq. (4) that the matrix $\mathbf{A}$ is the key point for determining the material parameters of the transformation media. According to polar decomposition, the deformation gradient tensor $\mathbf{A}$ can be decomposed into a pure stretch deformation (described by a positive definite symmetric tensor $\mathbf{V}$) and a rigid body rotation (described by a proper orthogonal tensor $\mathbf{R}$). Then we have $\mathbf{A} = \mathbf{VR}$ [20]. Suppose the material parameters in the original space are homogeneous and isotropic (for the simplicity, we let them equal to 1). Consider the left Cauchy–Green deformation tensor $\mathbf{B} = \mathbf{V}^2 = \mathbf{AA}^{\mathrm{T}}$ [20], from Eq. (4) we can obtain

$$\boldsymbol{\rho}'^{-1} = \mathbf{B}/\det\mathbf{A}. \tag{5}$$

In the principal system, the tensor $\mathbf{B}$ can be expressed by the diagonal components

$$\mathbf{B} = \mathrm{diag}[\lambda_1^2, \lambda_2^2, \lambda_3^2], \tag{6}$$

where λ_i (i = 1,2,3) are the eigenvalues of the tensor $\mathbf{V}$ or the principal stretches for an infinitesimal element. Using det $\mathbf{A} = \lambda_1\lambda_2\lambda_3$ and Eqs. (5), (6), we can rewrite Eq. (4) as

$$\boldsymbol{\rho}' = \mathrm{diag}\left[\frac{\lambda_2\lambda_3}{\lambda_1}, \frac{\lambda_3\lambda_1}{\lambda_2}, \frac{\lambda_1\lambda_2}{\lambda_3}\right], \quad \kappa' = \lambda_1\lambda_2\lambda_3. \tag{7}$$

Eq. (7) shows that the material parameters in the transformed space can be calculated by the principal stretches of the deformation during the space transformation. The material parameter tensors and the corresponding element deformation tensor have the same principal directions. It is also noted that the rigid body rotation has no contribution on the final transformed material parameters. Thus, the calculation of transformed material parameters is equivalent to the following mechanical problem: how to calculate the deformation fields for a given space transformation. If the principal stretches are obtained analytically from a space transformation, Eq. (7) can be used directly for designing the necessary material property; otherwise the numerical method for the deformation field must be used.

As an example of using Eq. (7) to determine the material parameters, consider a 2D cylindrical cloak formed by a linear transformation $r' = R_1 + r(R_2 - R_1)/R_2$, $\theta' = \theta$, and $z' = z$. The principal stretches for this transformation are $\lambda_r = dr'/dr = (R_2 - R_1)/R_2$, $\lambda_\theta = r'/r = r'(R_2 - R_1)/[(r' - R_1)R_2]$, and $\lambda_z = 1$. Substituting these expressions into Eq. (7), we can get

$$\rho'_r = \frac{\lambda_\theta}{\lambda_r} = \frac{r'}{r' - R_1}, \quad \rho'_\theta = \frac{\lambda_r}{\lambda_\theta} = \frac{r' - R_1}{r'},$$
$$\kappa' = \lambda_r\lambda_\theta = \left(\frac{R_2 - R_1}{R_2}\right)^2 \frac{r'}{r' - R_1}. \tag{8}$$

which coincide with results of Cummer and Schurig [14]. However, for an arbitrary shape cloak, it is very difficult to obtain the principal stretches analytically from the space transformation. In the following, we will propose a method to calculate the deformation field for an arbitrary shape cloak.

3. Acoustic cloak of arbitrary shape

For a given arbitrary cloak, the space transformation from the region Ω to Ω' is not unique, but the boundary condition for the cloak is predefined. Generally, the cloak is formed by expanding a point $\boldsymbol{O}$ in the region Ω to a new boundary $\partial\Omega'_-$, while keeping the outer boundary $\partial\Omega'_+$ unchanged during the transformation. Fig. 1 shows the scheme of forming a cloak. Theoretically, any object placed inside of the region $\partial\Omega'_-$ will be acoustically invisible to an outside observer. The pressure and velocity fields on the boundary of such constructed acoustic cloak have been discussed by Norris [17].

Now the crucial point for designing a cloak is how to calculate the deformation field within the cloak layer. The deformation field induced by the transformation must be continuous in order to have no reflection inside the cloak. It is known that the deformation is determined by the partial derivative of displacements in the transformed space with respect to the original coordinates [20]. Since the harmonic solution of the displacement field will always lead to continuous deformation, so we propose to use Laplace's equations with Dirichlet boundary conditions [21] to calculate the displacement field for cloaks. Here we will use the inverse form of Laplace's equations

$$\frac{\partial^2 x_i}{\partial x_1'^2} + \frac{\partial^2 x_i}{\partial x_2'^2} + \frac{\partial^2 x_i}{\partial x_3'^2} = 0, \quad i = 1, 2, 3; \quad x'_1, x'_2, x'_3 \in \Omega', \tag{9}$$

and the corresponding boundary conditions are

$$\mathbf{x}(\partial\Omega'_-) = 0, \quad \mathbf{x}(\partial\Omega'_+) = \mathbf{x}'(\partial\Omega'_+), \tag{10}$$

From Eqs. (9) and (10), the displacement field $\mathbf{x} = \mathbf{x}(\mathbf{x}')$ under the space transformation can then be obtained, the corresponding deformation field will be used to evaluate the material parameters of the cloak. For an arbitrary cloak, Laplace's equations need to be solved by mean of numerical method.

There are many numerical solvers for solving Laplace's equations. Here we use the commercial software COMSOL Multiphysics, since the determination of the material parameters and validation of the cloaking effect can be well integrated into a two-step model. The first step (step I) is the numerical calculation of the deformation field according to Laplace's Eq. (9) with the boundary condition Eq. (10). This step can be implemented by the PDE mode (Laplace's equations) of COMSOL Multiphysics. The second step (step II) is the numerical simulation to check the cloaking effect with the obtained material parameter in the step I by another PDE mode (Helmholtz equations) $\nabla \cdot (\mathbf{c}\nabla p) + ap = 0$, which has the same form as Eq. (1). In the two steps, the same geometrical object is used.

As an example, we consider a plane acoustic wave incident on a 2D cloak that's inner and outer boundaries are depicted *randomly*. Fig. 2 shows the computational domain for a horizontally incident

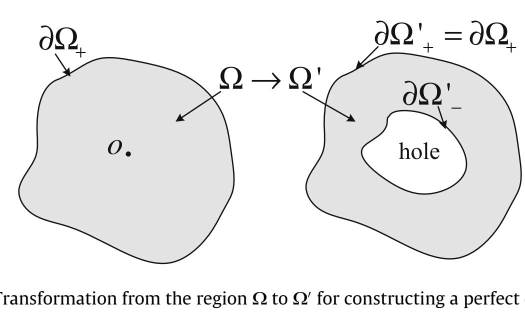

Fig. 1. Transformation from the region Ω to Ω' for constructing a perfect cloak. The outer boundary $\partial\Omega_+$ of the region Ω is kept unchanged, and the point $\boldsymbol{O}$ inside of Ω is transformed to the inner boundary $\partial\Omega'_-$ of Ω'. Except a point $\boldsymbol{O}$ and $\partial\Omega'_-$, the transformation is everywhere one-to-one and differentiable.

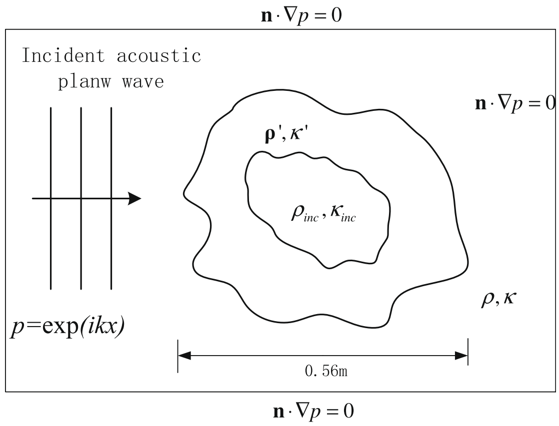

Fig. 2. Computational domain and simulation for a left-incident acoustic cloak based on Helmholtz equation.

wave. Fig. 3 shows the contour lines of the displacement by solving Laplace's equation with the boundary condition Eq. (10). The obtained displacement will be used to calculate the material parameters of the cloak. To verify the effect of the cloak, in the PDE mode of Helmholtz equation, we let a plane acoustic wave be incident on this cloak, so the incident boundary of the computation domain is set to be Dirichlet condition $p = \exp(-ikx)$ where $k = 2\pi/\iota$ is the wave number and ι is the wavelength. The other boundaries of the computation domain are set to be Neumann boundary conditions $\mathbf{n} \cdot \nabla p = 0$, in order to avoid disturbance for the wave propagation. In the following simulation, the background medium is water $\boldsymbol{\rho} = 1 \times 10^3$, $\kappa = 2.18 \times 10^9$ in SI units and the wavelength of the incident wave is $\iota = 0.35$ m. The material parameters within the cloak are $\boldsymbol{\rho}_{\text{inc}} = \boldsymbol{\rho}/5$ and $\kappa_{\text{inc}} = \kappa$, respectively. Fig. 4 shows the p field produced by these boundary conditions in the computation

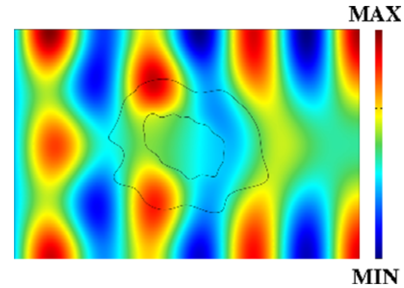

Fig. 5. Simulation of a left incident acoustic wave without the cloak layer.

domain without any obstacle. Fig. 5 shows the pressure field p scattered by the object without a cloak layer. It is seen that there are strong scattered fields outside of the cloak. Fig. 6 shows the pressure field of the same object with the designed cloak layer. It is clearly seen from the Fig. 6 that the designed arbitrary cloak doesn't disturb the outside field and shield the irregular obstacle inside of the cloak from detection, thus realizing acoustic *invisibility*. Compared to Fig. 4, the cloaking effect is not perfect; this imperfection is believed to be induced by the numerical method due to discretization. The necessary material parameters for the cloak retrieved from the calculation are shown in Fig. 7a–d for $1/\rho'_{xx}$, $1/\rho'_{xy}$, $1/\rho'_{yy}$ and the bulk modulus κ', respectively.

4. Discussion and conclusion

We have proposed a numerical method for design of an acoustic cloak with arbitrary shape. Laplace's equation is used to evaluate the deformation field induced by the transformation, the material parameters of the cloak can then be determined from the deformation field. Usually the transformation to construct a cloak is not unique, however, if the displacement field is supposed to be

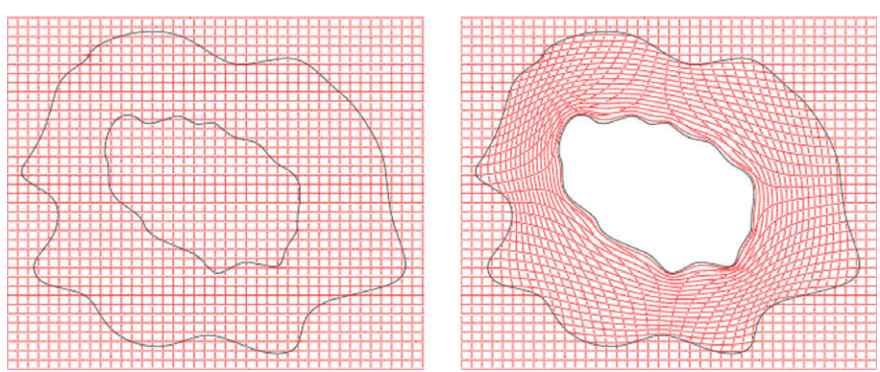

Fig. 3. Coordinate contour lines. Left: Original coordinate contour lines before transformation. Right: Original coordinate contour lines after the transformation.

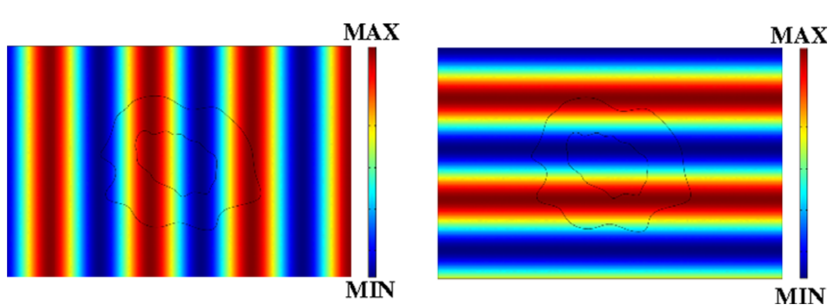

Fig. 4. Simulation of the acoustic wave without any obstacle. Left: left incident. Right: bottom incident.

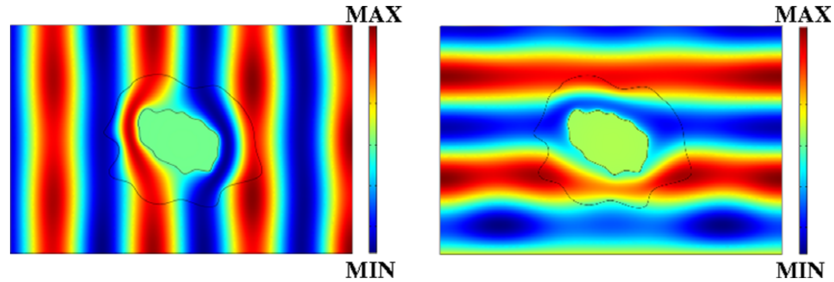

Fig. 6. Simulation of the acoustic wave with the designed cloak layer. Left: left incident. Right: bottom incident.

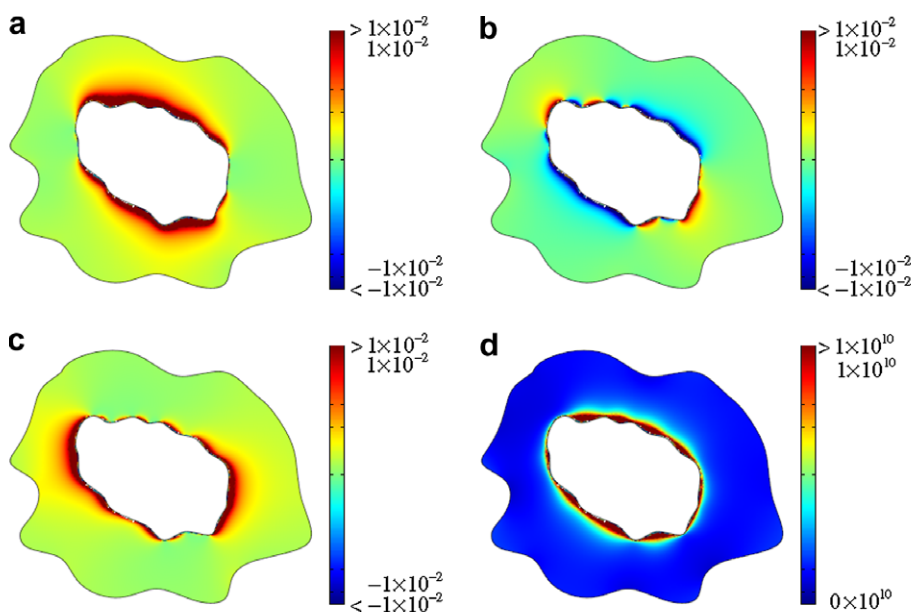

Fig. 7. Values of the non-vanishing components of the mass density tensors of the designed cloak, (a) $1/\rho'_{xx}$, (b) $1/\rho'_{xy}$, (c) $1/\rho'_{yy}$, and (d) bulk modulus κ'.

harmonic, it can be determined uniquely from the boundary condition. The ideal acoustic cloak requires an infinite mass in the inner boundary, which is unpractical [17], however, an impedance-matched reduced acoustic cloak is recently proposed in [22], this makes acoustic cloaks more realizable in practice. The acoustic metamaterials are often dispersive; the effect of cloaking is limited by bandwidth. Although a numerical example is given for a complicated cloak design, the proposed method is not limited to the numerical method. If analytical solutions can be found for Laplace's equation, we can get the analytical expression for the transformed material. Taking a spherical cloak as an example, we can solve Eq. (9) analytically in the spherical coordinates system by using the boundary conditions $r(a) = 0$ and $r(b) = $ b, then we get the transformation $r' = ab^2/[(a-b)r + b^2]$, where a and b are the radii of the outer and the inner boundaries of the cloak, respectively. This indicates a nonlinear transformation, different from the linear one $r' = a + r(b-a)/b$, initially proposed for a spherical acoustic cloak [12]. The corresponding principal stretches of this nonlinear transformation are respectively $\lambda_r = dr'/dr = ab^2(b-a)r'$, $\lambda_\theta = \lambda_\phi = r'/r = [(b-a)r'^2]/[b^2(r'-a)]$. Thus the materials parameters necessary for the spherical cloak can be obtained from Eq. (7) as

$$\rho'_r = \frac{\lambda_\theta\lambda_\phi}{\lambda_r} = [(b-a)r'^3]/ab^6(r'-a)^2, \quad \rho'_\theta = \rho'_\phi = \frac{\lambda_\phi\lambda_r}{\lambda_\theta} = ab^2(b-a)r',$$
$$\kappa' = \lambda_r\lambda_\phi\lambda_\theta = a(b-a)^3 r'^5/[b(r'-a)]^2. \tag{11}$$

In summary, we have proposed a method to evaluate the material parameters for an acoustic cloak with arbitrary shapes. The material parameters of the transformed material are calculated by the principle stretches of the deformation induced by the transformation. The functionality of the transformed material is converted to the boundary condition. With help of Laplace's equations, the deformation field within the cloak layer can be determined and further be used to compute the material parameters. The simulation result of the cloaking effect together with evaluation of the material parameters are well integrated in a two-step model with help of the finite-element software COMSOL Multiphysics. The proposed method can be easily extended to design other transformed acoustic media, such as concentrators and rotators.

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References

[1] D.R. Smith, J.B. Pendry, M.C.K. Wiltshire, Science 305 (2004) 788.
[2] D. Schurig, J.J. Mock, B.J. Justice, S.A. Cummer, J.B. Pendry, A.F. Starr, D.R. Smith, Science 314 (2006) 977.

[3] J.B. Pendry, D. Schurig, D.R. Smith, Science 312 (2006) 1780.
[4] H. Chen, C.T. Chan, Appl. Phys. Lett. 90 (2007) 241105.
[5] M. Rahm, D. Schurig, D.A. Roberts, S.A. Cummer, D.R. Smith, Photon. Nanostruct.: Fundam. Appl. 6 (2007) 87.
[6] M. Rahm, S.A. Cummer, D. Schurig, J.B. Pendry, D.R. Smith, Phys. Rev. Lett. 100 (2008) 063903.
[7] S. Yao, X. Zhou, G. Hu, New J. Phys. 10 (2008) 043020.
[8] A. Greenleaf, M. Lassas, G. Uhlmann, Commun. Pure Appl. Math. 56 (2003) 328.
[9] U. Leonhardt, Science 312 (2006) 1777.
[10] U. Leonhardt, New J. Phys. 8 (2006) 118.
[11] U. Leonhardt, New J. Phys. 8 (2006) 247.
[12] H. Chen, C.T. Chan, Appl. Phys. Lett. 91 (2007) 183518.
[13] S.A. Cummer, B. Popa, D. Schurig, D.R. Smith, J. Pendry, M. Rahm, A. Starr, Phys. Rev. Lett. 100 (2008) 024301.
[14] S.A. Cummer, D. Schurig, New J. Phys. 9 (2007) 45.
[15] A. Greenleaf, Y. Kurylev, M. Lassas, G. Uhlmann, Jan 2008. <http://arxiv.org/abs/0801.3279v1>.
[16] A. Greenleaf, Y. Kurylev, M. Lassas, G. Uhlmann, Comm. Math. Phys. 275 (3) (2007) 749.
[17] A.N. Norris, Proc. Roy. Soc. A 464 (2008) 2411.
[18] G.W. Milton, M. Briane, J.R. Willis, New J. Phys. 8 (2006) 248.
[19] J. Hu, X. Zhou, G. Hu, Calculation of material properties for arbitrary shape transformation media, in: International Workshop on Metamaterials, November 9–12, Nanjing, China, 2008.
[20] W.M. Lai, D. Rubin, E. Krempl, Introduction to Continuum Mechanics, third ed., Butterworth, Heinemann, Burlington, 1995.
[21] R. Courant, D. Hilbert, Methods of Mathematical Physics, first ed., vol. 2, Wiley Interscience, New York, 1989.
[22] H.Y. Chen, T. Yan, X.D. Luo, H.R. Ma, Chin. Phys. Lett. 25 (2008) 3696.