Controlling elastic waves with isotropic materials

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Design of functional devices with isotropic materials has significant advantages, as regards easy fabrication and broadband application. In this letter, we present a method to derive isotropic transformation material parameters for elastodynamics under local conformal transformation. The transformed material parameters are then applied to design a beam bender, a four-beam antenna and an approximate carpet cloak for elastic wave with isotropic materials, validated by the numerical simulations. © *2011 American Institute of Physics*. [doi:10.1063/1.3569598]

Transformation method provides a direct way to find material parameter distributions when wave propagation pattern is prescribed, and many interesting devices for electromagnetic (EM) waves are conceived[1–4] based on this technique. The method is also extended to acoustic wave for liquid materials, since the Helmholtz's equation is shown to have a transformation type solution.[5,6] An acoustic cloak has recently been demonstrated experimentally by Zhang *et al.*[7] Based on assumption of local affine transformation, Hu *et al.*[8] derived the transformed material parameters for elastic wave. Generally, the material derived by transformation method is anisotropic, thus some material parameters have to be realized with local resonant mechanism.[4,9,10] For easy fabrication and broadband application, devices with isotropic materials are in great demand. Along this vein, Leonhardt[2] proposed conformal mapping for transformation optics and acoustics with isotropic media in two-dimensional (2D) case. Li and Pendry[11] suggested quasi-conformal mapping to design a carpet cloak for EM waves; the proposed device can be made of isotropic materials, and this greatly simplifies the experimental implementations.[12,13] Conformal mapping is also applied to design directional antennas for EM wave[14] and acoustic wave[15] with isotropic materials. In this letter, we will explore this possibility for elastic wave. First, the transformed material parameters for elastic wave under local conformal mapping are derived; they are then applied to design a beam bender, a directional antenna, and an approximately carpet cloak with isotropic materials.

We start by considering an elastodynamic problem in a virtual space described by Navier's equation

$$\nabla \cdot \boldsymbol{\sigma} = \rho \cdot \ddot{\mathbf{u}}, \quad \boldsymbol{\sigma} = \mathbf{C}:\nabla \mathbf{u}, \tag{1}$$

where $\mathbf{u}$ denotes displacement vector, $\boldsymbol{\sigma}$ is second order stress tensor, $\mathbf{C}$ is fourth-order elasticity tensor and ρ means density. For a general spatial mapping $\mathbf{x}'=\mathbf{x}'(\mathbf{x})$ that transforms the virtual space to a physical space, Milton *et al.*[16] show that Navier's equation cannot keep its form. However, if we consider the transformation in a local view and adopt locally an affine transformation point by point, the transformed governing equation can still keep its form, i.e., Navier's equation, and the transformed material parameters for elastic wave can be derived.[8] With assumption of local affine transformation for a mapping, the deformation gradient tensor induced by the mapping $\mathbf{A}=\nabla_{\mathbf{x}}\mathbf{x}'$ can be decomposed as $\mathbf{A}=\mathbf{VR}$, where $\mathbf{R}$ and $\mathbf{V}$ denote a rigid rotation and a pure stretch tensor, respectively.[17] A local Cartesian frame $\mathbf{e}'$ at a point $\mathbf{x}'$ is established in the physical space, which is the principal frame of $\mathbf{V}$ $(\mathbf{V}=\lambda_1\mathbf{e}_1'\otimes\mathbf{e}_1'+\lambda_2\mathbf{e}_2'\otimes\mathbf{e}_2'+\lambda_3\mathbf{e}_3'\otimes\mathbf{e}_3')$. During the mapping, the stress, displacement, stiffness and density attached on a spatial element at the point $\mathbf{x}$ in the virtual space will experience a rigid rotation $\mathbf{R}$ and then scaling along $\mathbf{e}'$ to reach the point $\mathbf{x}'$ in the physical space. Symbolically the transformations can be written as[8]

$$\mathbf{V_q}\mathbf{R}:\mathbf{q}\mapsto\mathbf{q}', \quad \mathbf{q}=\boldsymbol{\sigma},\mathbf{u},\mathbf{C},\boldsymbol{\rho}, \tag{2}$$

where $\mathbf{V_q}$ is the scaling tensor for the quantity $\mathbf{q}$, and has a diagonal form in the specially established frame $\mathbf{e}'$, i.e.,

$$\mathbf{V}_{\boldsymbol{\sigma}}=\mathrm{diag}[a_1,a_2,a_3],\mathbf{V_u}=\mathrm{diag}[b_1,b_2,b_3],\mathbf{V_C} =\mathrm{diag}[c_1,c_2,c_3],\mathbf{V}_{\boldsymbol{\rho}}=\mathrm{diag}[d_1,d_2,d_3], \tag{3}$$

where a_i, b_i, c_i, and d_i are scaling factors for stress, displacement, stiffness, and density, respectively, which are to be determined. With local affine transformation, the form-invariance of Navier's equation and conservation of energy at each point lead to the following conditions for determining the scaling factors[8]

$$\frac{a_ia_j}{d_Jb_J}=\lambda_i, \quad \frac{a_ia_j}{c_Ic_Jc_Kc_lb_k}=\frac{1}{\lambda_l}, \quad a_ia_jb_I=\frac{\lambda_j}{\lambda_1\lambda_2\lambda_3}. \tag{4}$$

The capital letter in the index means the same value as its lower case, but without summation.

We further consider a local conformal mapping at each point, i.e., $\lambda_1=\lambda_2=\lambda_3\equiv\lambda$, then the scaling tensor $\mathbf{V_q}$ becomes isotropic, and Eq. (4) becomes

$$\frac{a^2}{bd}=\lambda, \quad \frac{a^2}{c^4b}=\frac{1}{\lambda}, \quad a^2b=\frac{1}{\lambda^{N-1}}, \tag{5}$$

where $N=2,3$ for 2D and three-dimensional problems, respectively. Obviously, there are nonunique solutions for the scaling factors a, b, c, and d when λ is given. We can express three of them in terms of the rest one and λ for example, $b=1/(a^2\lambda^{N-1})$, $c^4=a^4\lambda^N$, $d=a^4\lambda^{N-2}$. Since the density ρ and the modulus tensor $\mathbf{C}$ in the virtual space are isotropic, the transformed material parameters are finally derived in the global frame as

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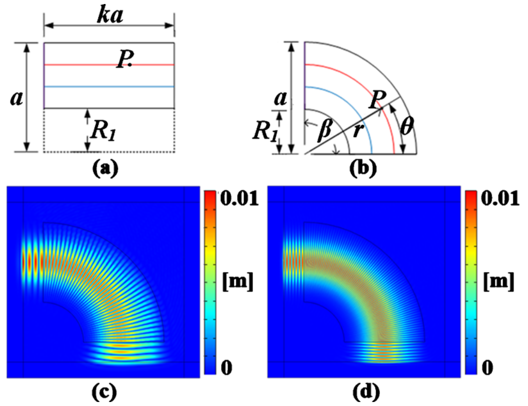

FIG. 1. (Color online) Transformation for a beam bender: (a) virtual space and (b) physical space. Total displacement $\sqrt{u_x^2+u_y^2}$ for (c) P-wave and (d) S-wave in the bender.

$$\mathbf{C}' = a^4\lambda^N\mathbf{C}, \rho' = a^4\lambda^{N-2}\rho, \boldsymbol{\sigma}' = a^2\mathbf{R}\boldsymbol{\sigma}\mathbf{R}^{\mathbf{T}}, \mathbf{u}' = a^{-2}\lambda^{1-N}\mathbf{R}\mathbf{u}. \tag{6}$$

Equation (6) provides the transformed material parameters for a local conformal transformation, and both modulus and density are isotropic in the physical space. We shall apply the transformed material parameters to design three functional devices with isotropic materials in the following: a beam bender, a directional antenna and an approximate carpet cloak.

A 2D beam bender is designed by a mapping that transforms a rectangular plate into a plate of an arc shape. The conformal transformation is made by setting the stretch along $\hat{\mathbf{r}}$ equal to that along $\hat{\boldsymbol{\theta}}$ at each point, as used to design isotropic EM bender,[18] and the stretches are finally given by

$$\lambda_r = \lambda_\theta = \lambda = \beta r/(ka), \tag{7}$$

where β is polar angle, a is original length of the rectangular plate and k has arbitrarily nonzero real value, as shown in Figs. 1(a) and 1(b). Different from EM wave, special care must be taken to assure the impedance-matched condition between the transformed and the untransformed regions. For perpendicularly incident waves, the impedance-matched condition for both S and P waves is given as[19]

$$\rho'\mathbf{C}' = \rho\mathbf{C}. \tag{8}$$

With help of Eq. (6), the impedance-matched transformed material parameters are derived as $\mathbf{C}'=\lambda\mathbf{C}$, $\rho'=\rho/\lambda$, or

$$E' = \lambda E, \quad \upsilon' = \upsilon, \quad \rho' = \rho/\lambda, \tag{9}$$

where E and υ denote Young's modulus and Poisson's ratio of the media, respectively. To validate the transformed material parameters for the bender, numerical simulations are performed with the structural mechanics module of finite element method software COMSOL MULTIPHYSICS, where a beam of a P-wave or S-wave is incident on the bender. The background is the structural steel with material parameters $E=200$ GPa, $\upsilon=0.33$, and $\rho=7850$ kg/m^3. The simulation results on wave propagation pattern are shown in Figs. 1(c) and 1(d) with frequency 20 kHz and $\beta=\pi/2$, $a=3$ m, $k=\pi/3$ for P-wave and S-wave, respectively. They show clearly that the waves are guided to a new direction as desired.

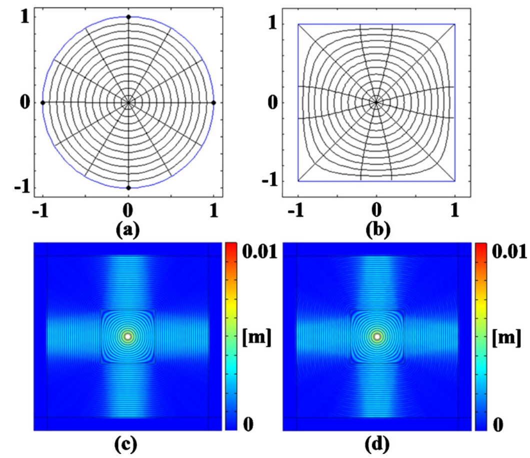

FIG. 2. (Color online) Transformation for the directional four-beam antenna: (a) virtual space and (b) physical space. Total displacement $\sqrt{u_x^2+u_y^2}$ for (c) P-wave and (d) S-wave incident from the center of the antenna.

Directional four-beam antenna for elastic wave with isotropic material can be realized by Schwartz–Christoffel conformal transformation, defined by[14,15]

$$w = \alpha \cdot 2i \cdot F\left\{ i \cdot \sinh^{-1}\left[\left(i\frac{1+z}{1-z} - 1 \right)^{-1/2} \right] \middle| 2 \right\} + \gamma, \tag{10}$$

where $w=x'+iy'$, $z=x+iy$, and $F[\varphi|m]$ denotes the incomplete elliptic integral of the first kind with modulus m, the coefficients $\alpha=-(2+2i)/[2K(-1)+i\sqrt{2}K(1/2)]$, $\gamma=-1+i$, where $K[\varphi]$ is the complete elliptic integral of the first kind. This mapping converts a cylindrical wave into plane ones. Once the transformation [shown in Figs. 2(a) and 2(b)] is provided, the local principal stretch λ can be then evaluated, and the materials to realize this function are obtained through the transformations given by Eq. (9). In the numerical simulation, the conformal transformation is generated using MATLAB toolbox developed by Driscoll.[20] A line source of P-wave or S-wave is generated in the center of the antenna. The background is the same as that used in the beam bender. During the conformal transformation, singularities are formed at the intersection of the edges of the device, and they are set to a finite large value in the numerical simulation, as in the references.[14,15] Figures 2(c) and 2(d) show the wave patterns for the P-wave and S-wave, respectively. Indeed, a cylindrical wave is transformed into four plane wave beams, as designed. For acoustic wave, the stiffness tensor is characterized only by bulk modulus, and the transformed material parameters proposed by Ren *et al.*[15] can be obtained as a special case.

Finally, we will apply the isotropic material parameters to design approximately an elastic carpet cloak. The method proposed by Chang *et al.*[21] is used to design the elastic carpet cloak. Since the stretches and impedance-mismatch near the upper and side boundaries of the cloak are very small, we can further simplify the material parameter by assuming constant density or constant modulus. For constant density, we have $a^4=1/\lambda^{N-2}$ from Eq. (6), and the following transformed material parameters can be derived:

$$E' = \lambda^2 E, \quad \upsilon' = \upsilon, \quad \rho' = \rho. \tag{11a}$$

For constant modulus, we get

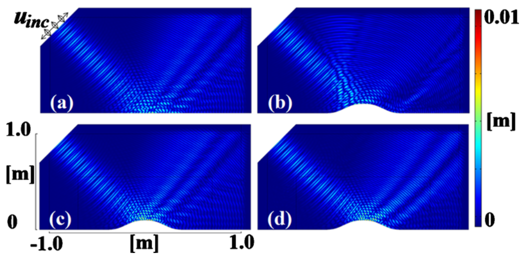

FIG. 3. (Color online) Wave (both P-wave and S-wave) incidents on the (a) linear and (b) curvilinear boundaries without the carpet cloak. Wave incidents on the curvilinear boundaries with the (c) isotropic and (d) anisotropic carpet cloak.

$$E' = E, \quad \upsilon' = \upsilon, \quad \rho' = \rho/\lambda^2. \tag{11b}$$

In order to validate the transformed material parameters, an elastic carpet cloak is designed by using Eq. (11a). The background is set to be the same as that of the beam bender. Harmonic waves including both P-wave and S-wave are emitted from a line source, as shown in Fig. 3(a). Figure 3(c) shows that through the isotropic carpet cloak, the scattering of an incident wave by a curvilinear boundary is the same as that by a linear boundary [Fig. 3(a)] if observed outside the cloak. Compared to the anisotropic elastic carpet cloak[8] shown in Fig. 3(d), the approximate isotropic carpet cloak works perfectly well. In our example, the anisotropy factor[11] is about 1.07. The same result can also be found for a carpet cloak designed with Eq. (11b). It is worthwhile to point out that the curvilinear boundary not only affects the reflection direction, but also the mode of the reflected wave due to the mode conversion at the boundary.[19] However, the designed carpet cloak can restore both the mode and the propagation direction.

To conclude, the isotropic transformed material parameters for elastic wave under local conformal mapping are derived, which are then applied to design some interesting devices for elastic wave with isotropic materials, including a beam bender, a four-beam antenna. We also show that the isotropic transformation material parameters can also be applied to design carpet cloak with a good approximation.

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